Mathematical Properties and Financial Applications of a Novel Mean-reverting Random Walk in Discrete and Continuous Time

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Abstract. Mean-reversion is an essential characteristic of some financial time series including interest rates, commodity prices, and commodity spreads. Commodity prices, and certain special spreads between them, are often described by the simple and analytically tractable Vasicek model. However, the Vasicek model is inadequate to describe all commodity price series. We introduce a new family of one dimensional stochastic processes built from a mean-reverting random walk on a lattice. We then obtain some analytical results about its solution including its stationary distribution. This new mean-reverting process is compared with the Vasicek process and its advantages are discussed.

Keywords: Mean-reversion, Mean-reverting random walk, Vasicek process, Mean-reverting stochastic process.

1 Introduction

Understanding commodity prices is one of the final frontiers of quantitative finance. Commodity markets facilitate the risk management of trading in consumption goods such as agricultural commodities (eg. cotton, wheat, soybeans), mineral commodities (eg. copper, zinc, lead), and energy commodities (eg. electricity, oil, and natural gas). The goods are obtained by costly and difficult work from farms, mines, or oil wells. As such, increased prices tend to attract more producers to the marketplace, while decreased prices tend to drive producers from the market place. The resulting supply/demand dynamics often cause mean reverting behavior in markets for commodity prices.

In a related way, the spreads between commodity prices are often interesting as well. For example, corn may be converted to ethanol via a chemical reaction

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(See Kirby and Davison [6] for more detail). Natural gas may be converted to electrical power by running a gas turbine. The resulting spreads have names: the spread between the corn and the ethanol price is called the crush spread while the spread between the gas and the electricity price is called the spark spread. Because plants can be run at a profit but idled at a loss, the time series describing these spread processes are also often mean reverting.

A simple continuous time model for prices is known as the Ornstein-Uhlenbeck process. Vasicek adapted this process to the study of interest rates 1997 [9]. The model has the form:

$$dX_t = \kappa\,(\mu - X_t)\ dt + \sigma\, dW_t, \tag{1}$$

and describes the evolution of a one dimensional rate (or price) over time. This model is simple and analytically tractable, but is not appropriate to model all time series. In this paper we propose another model for a one dimensional mean reverting stochastic process. Our model is built up from a random walk in which a price can take on values on a spatial lattice related to the integers and makes time steps on a time lattice indexed by the natural numbers. It is interesting to connect price processes to random walks as they can provide nice intuition.

A random walk (RW) is a mathematical mechanism to model a path based on a succession of random steps. Random walks are deployed in many sciences such as finance, physics, economics, and computer science to capture behaviors of various processes [10]. A random walk can be applied to trace the behavior of various paths including the evolution of stock prices, the financial status of a gambler, a drunkard walking, and a molecule traveling in liquid [8]. Random walks have various forms and are often considered as Markov chains [1]. Random walks can occur in one, two, or many dimensions. Moreover, random walks may have different time dynamics. For example, simple discrete time walks are indexed by the natural numbers while some sophisticated walks are assumed to take steps at random times [5].

Here, we construct a walk that is mean-reverting. In this mean-reverting random walk (MRW), we define the probabilities of taking each step either forward or backward to depend on the current location of the walker. When the walker diverges from the mean, by changing the probabilities of traveling forward and backward, the walker will tend to revert back to the mean, which means that the restoring force will be stronger as the walker deviates further away from the mean. This paper is organized as follows:
In Section 2, we define our novel mean-reverting random walk. In Section 3, we derive the *scaling limit* of this mean-reverting RW and find its stochastic process. In Section 4 and 5, we develop analytical results about the continuous time problem. The new mean-reverting process is compared to the Vasicek process [9] and its advantages are discussed in Section 6.

2 Mean-reverting Random Walk:

To begin, consider a random walk defined on an integer lattice. At time steps indexed by the natural numbers the walker moves either a step to the right (moving a distance of +1) or a step to the left (moving a distance of -1). We define the position of the walker immediately after the k^{th} time step by S_k. We assume that the walker begins at the origin; in other words, $S_0 = 0$.

Now, let P_i denote the probability of moving right ($S_{i+1} - S_i = 1$) at the i^{th} step. We make this probability depend on the location of the walker as follows:

$$P_i(\text{right}) = \begin{cases} \dfrac{1}{2 + aS_{i-1}}, & \text{if } S_{i-1} \geq 0 \\ 1 - \dfrac{1}{2 - aS_{i-1}}, & \text{elsewhere} \end{cases} = \begin{cases} \dfrac{1}{2} - \dfrac{aS_{i-1}}{2(2 + aS_{i-1})}, & \text{if } S_{i-1} \geq 0, \\ \dfrac{1}{2} - \dfrac{aS_{i-1}}{2(2 - aS_{i-1})}, & \text{elsewhere,} \end{cases} \tag{2}$$

where
$S_{i-1} \in \mathbb{Z}$ is the current location,
$a \geq 0$ is the mean-reversion speed.

As S_{i-1} increases the probability of going right decreases and vice versa. When S_{i-1} approaches infinity, P_i approaches zero. As a decreases, positive P_i decreases or negative P_i increases with lower speed. The dynamics described by equation 2 approaches a simple random walk as a approaches to zero. Denote by n the number of steps taken: each step is either $+1$ or -1 [3]. Then the number of possible different paths that can be traveled will be 2^n. The number of walks that satisfy $S_n = k$ where $k > 0$ equal to the number of ways of choosing $(n+k)/2$ elements from an n element set (for this to be non-zero, it is necessary that $n+k$ be an even number), $\binom{n}{(n+k)/2}$. Note that for simple random walk the $P(S_n = k)$ is equal to $2^{-n}\binom{n}{(n+k)/2}$. Here, in this new setting, the probability changes according to the location; as a result, each path to reach k from the origin in n walks has its own probability and may be different from another path with the same endpoints. One evident (but somewhat impractical) way to calculate this probability is to calculate the probability for every possible path and sum them up to determine the $P(S_n = k)$. The following Lemma proves the symmetry of this walk process.

Lemma 21 The mean-reverting random walk generated by the transition probability function 2 is symmetric:

We need to show that $P(S_n = k) = P(S_n = -k)$ where $k > 0$. To show this, it suffices to prove that, for any path $R = \{0 \to k_1 \to k_2 \cdots \to k_{n-1} = k-1 \to k\}$ that reaches k starting from the origin, there exists a corresponding path, $R^- = \{0 \to -k_1 \to -k_2 \cdots \to k_{n-1} = -k+1 \to -k\}$ to reach $-k$ starting from the origin with identical probability. To build path R^- from path R, we start from the origin and, at each step of path R, we take a step for R^- in the opposite direction. By continuing this method at any given stage of building the paths, the locations in the paths have identical distances but opposite direction *from the origin. Clearly, based on the definition of the transition probability function 2, the probabilities of path R and R^- are equal.*

3 Mathematical Derivation of Continuous-time Form

We attempt to derive the probability of being in location k after taking n steps, $P(k\,,\,n)$. We obtain the difference equation to calculate $P(k\,,\,n)$ for $k \in \{-n, -n+2, \cdots, n-2, n\}$ as follows:

$$P(k\,,\,n) = \begin{cases} \left(\dfrac{1}{2} + \dfrac{a(k+1)}{2(2\,+\,a(k\,+\,1))}\right) P(k+1\,,\,n-1) + \\ \quad \left(\dfrac{1}{2} - \dfrac{a(k-1)}{2(2\,+\,a(k\,-\,1))}\right) P(k-1\,,\,n-1), & \text{if } k \geq 1, \\ \left(\dfrac{1\,+\,a}{2\,+\,a}\right) P(1\,,\,n-1) + \\ \quad \left(\dfrac{1\,+\,a}{2\,+\,a}\right) P(-1\,,\,n-1), & \text{if } k = 0, \\ \left(\dfrac{1}{2} + \dfrac{a(k+1)}{2(2\,-\,a(k\,+\,1))}\right) P(k+1\,,\,n-1) + \\ \quad \left(\dfrac{1}{2} - \dfrac{a(k-1)}{2(2\,-\,a(k\,-\,1))}\right) P(k-1\,,\,n-1), & \text{if } k \leq -1, \end{cases} \tag{3}$$

where

$$\begin{cases} P(0\,,\,0) \;=\; 1, \\ P(k\,,\,0) \;=\; 0 \quad \text{for } k \neq 0. \end{cases}$$

One way to understand the dynamics described by equation 3 is to transform it to continuous form and work to understand the corresponding partial differential equation (PDE) using powerful tools of mathematical analysis. To do so, we define step size as Δx taking each Δt time. But before we proceed, we must modify the probabilities of moving right or left in equation 2 such that to be applicable in continuous form as follows:

$$P(\text{right}) = \begin{cases} \dfrac{1}{2} - \dfrac{ax}{2(2\,+\,ax)}\dfrac{\Delta t}{\Delta x}, \text{ if } x \geq 0, \\ \dfrac{1}{2} - \dfrac{ax}{2(2\,-\,ax)}\dfrac{\Delta t}{\Delta x}, \text{ elsewhere}, \end{cases} \tag{4}$$

Note that at given arbitrary location x, the probability of moving right, $P_r = \frac{1}{2} - \frac{ax}{2(2\,+\,a|x|\,)}$ and the probability of moving left, $P_l = \frac{1}{2} + \frac{ax}{2(2\,+\,a|x|\,)}$ lead to an average drift of $P_r - P_l = \frac{-ax}{2\,+\,a|x|}$ space steps per time step. In other words, this leads to an average speed of $\left\{\frac{-ax}{2\,+\,a|x|}\,\frac{\Delta x}{\Delta t}\right\}$. That is all fine as long as we keep the time and space steps the same. But it leads to difficulties when we refine the grid by taking the limit as $\Delta x \to 0$ and $\Delta t \to 0$. We observe that we want to make the mean speed the same no matter what time step we pick. To get that we write $P_r = \frac{1}{2} - \frac{ax\Delta t}{2\Delta x(2+a|x|)}$ and similarly for $P_l = \frac{1}{2} + \frac{ax\Delta t}{2\Delta x(2+a|x|)}$. This will give us a net speed of $\frac{-ax\Delta t}{\Delta x(2+a|x|)}\frac{\Delta x}{\Delta t} = \frac{-ax}{2+a|x|}$, independent of the discretization. In the original integer lattice setting, we do not encounter this point because $\Delta t = \Delta x = 1$, so it does not make any difference. This mean-reverting random walk 4 resembles a single random walk with drift. The probability that a particle is at location $x = k\Delta x$ at time $t = n\Delta t$, where $k \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$ is:

$$P(x\,,\,t) = \begin{cases} \left(\frac{1}{2} + \frac{a(x\,+\,\Delta x)}{2(2\,+\,a(x\,+\,\Delta x))}\frac{\Delta t}{\Delta x}\right)P(x+\Delta x\,,\,t-\Delta t) + \\ \quad\left(\frac{1}{2} - \frac{a(x\,-\,\Delta x)}{2(2\,+\,a(x\,-\,\Delta x))}\frac{\Delta t}{\Delta x}\right)P(x-\Delta x\,,\,t-\Delta t), \text{ if } x > 0, \\ \left(\frac{1}{2} + \frac{a(x\,+\,\Delta x)}{2(2\,-\,a(x\,+\,\Delta x))}\frac{\Delta t}{\Delta x}\right)P(x+\Delta x\,,\,t-\Delta t) + \\ \quad\left(\frac{1}{2} - \frac{a(x\,-\,\Delta x)}{2(2\,-\,a(x\,-\,\Delta x))}\frac{\Delta t}{\Delta x}\right)P(x-\Delta x\,,\,t-\Delta t), \text{ if } x < 0, \end{cases} \tag{5}$$

Equivalently, equation 5 can be written in the following general form for arbitrary $x \in \Re$:

$$\begin{aligned} P(x\,,\,t) = &\left(\frac{1}{2} + \frac{a(x\,+\,\Delta x)}{2(2\,+\,a\mid x\,+\,\Delta x\mid)}\frac{\Delta t}{\Delta x}\right)P(x+\Delta x\,,\,t-\Delta t) \,+ \\ &\left(\frac{1}{2} - \frac{a(x\,-\,\Delta x)}{2(2\,+\,a\mid x\,-\,\Delta x\mid)}\frac{\Delta t}{\Delta x}\right)P(x-\Delta x\,,\,t-\Delta t), \end{aligned} \tag{6}$$

To calculate $P(x\,,\,t{+}\Delta t)$ for arbitrary ($x \in \Re$), we deploy equation 6 as follows:

$$\begin{aligned} P(x\,,\,t+\Delta t) = &\left(\frac{1}{2} + \frac{a(x\,+\,\Delta x)}{2(2\,+\,a\mid x\,+\,\Delta x\mid)}\frac{\Delta t}{\Delta x}\right)P(x+\Delta x\,,\,t) \,+ \\ &\left(\frac{1}{2} - \frac{a(x\,-\,\Delta x)}{2(2\,+\,a\mid x\,-\,\Delta x\mid)}\frac{\Delta t}{\Delta x}\right)P(x-\Delta x\,,\,t) \end{aligned} \tag{7}$$

or equation 7 can be written in the form as follows:

$$\begin{aligned} P(x\,,\,t+\Delta t) = &\,\frac{1}{2}\left[\,P(x+\Delta x\,,\,t) \,+\, P(x-\Delta x\,,\,t)\right] \,+ \\ &\frac{\Delta t}{2\,\Delta x}\left[\,f(x+\Delta x\,,\,t) \,-\, f(x-\Delta x\,,\,t)\right], \end{aligned} \tag{8}$$

where $f(x\,,\,t) = \left(\frac{ax}{2+a|x|}\right) P(x\,,\,t)$.

Expand all terms on both side of equation 8 in a Taylor series as follows:

$$P(x\,,\,t+\Delta t) = P(x,t) \,+\, \frac{\partial P(x,t)}{\partial t}\Delta t \,+ \frac{1}{2}\,\frac{\partial^2 P(x,t)}{\partial t^2}\Delta t^2 \,+\, O(\Delta t^3),$$

$$\frac{1}{2}\left[\,P(x+\Delta x\,,\,t) \,+\, P(x-\Delta x\,,\,t)\right] \,=\, P(x,t) + \frac{1}{2}\,\frac{\partial^2 P(x,t)}{\partial x^2}\Delta x^2 \,+\, O(\Delta x^4),$$

$$\frac{\Delta t}{2\,\Delta x}\left[\,f(x+\Delta x\,,\,t) \,-\, f(x-\Delta x\,,\,t)\right] = \frac{\Delta t}{\Delta x}\left\{\frac{\partial f(x,t)}{\partial x}\Delta x \,+\, O(\Delta x^3)\right\}, \quad (9)$$

Insert equation 9 into equation 8 and after simplifying and dividing both sides by Δt , we have:

$$\frac{\partial P}{\partial t} \,+\, \frac{\partial^2 P}{2\partial t^2}\Delta t \,+\, \cdots \,=$$

$$\frac{\partial}{\partial x}\left\{\frac{ax}{2\,+\,a\mid x\mid}P\right\} \,+\, \frac{1}{2}\frac{\partial^2}{\partial x^2}\left\{P\frac{\Delta x^2}{\Delta t}\right\} \,+\, \cdots \qquad (10)$$

For this approach to work, we must take the limit in equation (10) as $\Delta x \to 0$ and as $\Delta t \to 0$ in a particular way such that:

$$D \,=\, \lim_{\Delta x, \Delta t \to 0} \frac{\Delta x^2}{\Delta t}, \quad \text{some number } D.$$

This is the same continuous limit taken when deriving the diffusion equation from the simple "drunkards walk" (see Davison 2014 [3]). Therefore, the resulting PDE for arbitrary $x \in \Re$ is:

$$\frac{\partial P(x\,,\,t)}{\partial t} \,=\, -\frac{\partial}{\partial x}\left(\frac{-ax}{2\,+\,a\mid x\mid}P(x\,,\,t)\right) \,+\, \frac{D}{2}\,\frac{\partial^2 P(x\,,\,t)}{\partial x^2}. \qquad (11)$$

It is now easy to generalize the PDE in equation 11 by including another parameter $\kappa > 0$, to have another control for mean-reverting speed for arbitrary $x \in \Re$, as follows (Later in Section 5, we will show a way to reparameterize this three parameter system back to two slightly different parameters):

$$\frac{\partial P(x\,,\,t)}{\partial t} \,=\, -\frac{\partial}{\partial x}\left(\frac{-\kappa ax}{2\,+\,a\mid x\mid}P(x\,,\,t)\right) \,+\, \frac{D}{2}\,\frac{\partial^2 P(x\,,\,t)}{\partial x^2}, \qquad (12)$$

where the boundary conditions and the initial condition are as follows:

$$\begin{aligned} \lim_{x\to\pm\infty} P(x\,,\,t) = \lim_{x\to\pm\infty} \frac{\partial P(x\,,\,t)}{\partial x} \,&=\, 0, \\ P(x\,,\,t=0) = \delta(x), & \end{aligned} \qquad (13)$$

and $\delta(x)$ is the *Dirac delta* function $\left(\delta(x) = \lim_{\epsilon\to 0}\frac{1}{\epsilon\sqrt{2\pi}}\exp\left(\frac{-x^2}{2\epsilon^2}\right)\right)$.

3.1 The Fokker Planck Equation

The *Fokker Planck equation (FPE)* provides a practical methods for stochastic modeling in wide range of studies including finance, physics, and biology [7]. The FPE describes the probability density function that evolves in time under a given continuous stochastic process as a partial differential equation. The general one-dimension FPE for the probability density function, $P(x\,,\,t)$ is in the following generic form:

$$\frac{\partial P(x\,,\,t)}{\partial t} \;=\; -\frac{\partial}{\partial x}\left[\mu(x,t)P(x\,,\,t)\right] \,+\, \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[D(x,t)P(x\,,\,t)\right], \tag{14}$$

where $\mu(x,t)$ is the drift or force and $D(x,t)$ is the diffusion coefficient.
The stochastic differential equation (SDE) for this $P(x\,,\,t)$ takes the form:

$$dX_t = \mu(X_t,t)\,dt + \sqrt{D(X_t,t)}\,dW_t. \tag{15}$$

Therefore, the equivalent SDE for our FPE in equation 12 is given:

$$dX_t = \frac{-\kappa a X_t}{2\,+\,a\mid X_t\mid}\,dt + \sigma\,dW_t, \tag{16}$$

where κ, a and $\sigma > 0$, and dW_t is the increment of a standard Brownian motion. The SDE in equation 16 is clearly mean-reverting since the drift always has the opposite sign to that of the location of the particle. The drift also increases with the distance of the particle to the origin, although it reaches a limit. Figure 1 depicts comparison between simulated probability density functions of X_t for the SDE in equation 15 for given parameters. By examining these graphs, we see that, as we decrease a, the distribution approaches to normal and as we increase a, the distribution has higher peak and thinner tails as compared to a normal density. Since a larger a describes more mean reversion, this is what we would expect.

4 The Stationary Solution

Useful insight about a stochastic process can be gleaned by determining the steady state, or stationary, behavior.

Definition 41 Stationary Solution (Distribution): *A unique solution to the FPE 14 or its equivalent SDE 15 is called a stationary distribution and is denoted by $P_{st}(x)$ if the limiting distribution of X_T as $T \to +\infty$ exists:*

$$\lim_{t\to+\infty} P(x\,,\,t) = P_{st}(x). \tag{17}$$

We should note that in FPE 14 or its equivalent SDE 15, if the process has a stationary solution (distribution), $\mu(x,t)$ and $D(x,t)$ are independent of t but not necessarily vice versa.

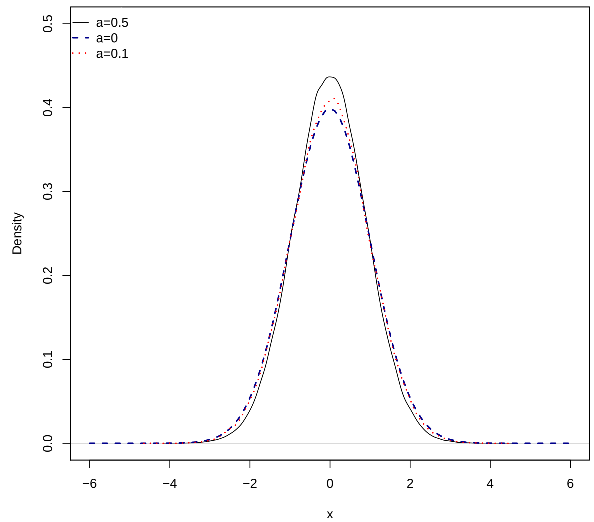

Fig. 1. Comparison between simulated probability density functions of X_t for the SDE in equation 15 for given parameters. Here, for all these three random walks, we assumed $\kappa = 1$, $\sigma = 1$, $\Delta t = 1/252$, $T = 1$ and $100,000$ simulated paths.

We attempt to derive a stationary solution for the mean-reverting process defined in equation 12 (or equivalently in equation 16). Based on the definition of a stationary solution 17 and from equation 12, we have:

$$-\frac{d}{dx}\left[\frac{-\kappa a x}{2\,+\,a\mid x\mid}P_{st}(x)\right]\,+\,\frac{\sigma^2}{2}\,\frac{d^2P_{st}(x)}{dx^2}=0. \tag{18}$$

Or

$$-\frac{d}{dx}\left[\frac{-\kappa a x}{2\,+\,a\mid x\mid}P_{st}(x)\,-\,\frac{\sigma^2}{2}\,\frac{dP_{st}(x)}{dx}\right]=0. \tag{19}$$

Based on the boundary conditions, we will have zero flux:

$$\frac{-\kappa a x}{2\,+\,a\mid x\mid}P_{st}(x)\,-\,\frac{\sigma^2}{2}\,\frac{dP_{st}(x)}{dx}=c, \qquad \text{with} \quad c=0\,(\text{by applying BC (13)}) \tag{20}$$

Therefore, we have:

$$\frac{dP_{st}(x)}{P_{st}(x)}=\frac{-2\kappa a x}{\sigma^2(2\,+\,a\mid x\mid)}dx. \tag{21}$$

Integrating equation 21 yields the solution as follows:

$$P_{st}(x)=\begin{cases} c^{-1}\exp\left(\frac{4\kappa\ln(2+ax)}{a\,\sigma^2}-\frac{2\kappa x}{\sigma^2}\right), & \text{if } x\geq 0,\\ \\ c^{-1}\exp\left(\frac{4\kappa\ln(2-ax)}{a\,\sigma^2}+\frac{2\kappa x}{\sigma^2}\right), & \text{otherwise,}\end{cases} \tag{22}$$

where the normalization constant c is obtained from:

$$c = 2\int_0^{\infty} \exp\left(\frac{4\kappa \ln(2+ax)}{a\,\sigma^2} - \frac{2\kappa x}{\sigma^2}\right) dx,$$

It turns out that this integral may be written in term of the exponential integral function, which is defined as $E_n(z) = \int_1^{\infty} \frac{e^{-zt}}{t^n} dt$ as follows:

$$c = \frac{2^{2+\frac{4\kappa}{a\,\sigma^2}}\, e^{\frac{4\kappa}{a\,\sigma^2}}}{a} E_{\frac{-4\kappa}{a\,\sigma^2}}\left(\frac{4\kappa}{a\,\sigma^2}\right), \tag{23}$$

Figure 2 shows the stationary solutions (distributions) for the MRW given in SDE form in equation 15 with stationary solution is in equation 22. By examining these graphs, we can see that as a increases, the stationary distribution has a higher peak and thinner tails. The long-run mean and the skewness of the stationary solution 22 are both zero.

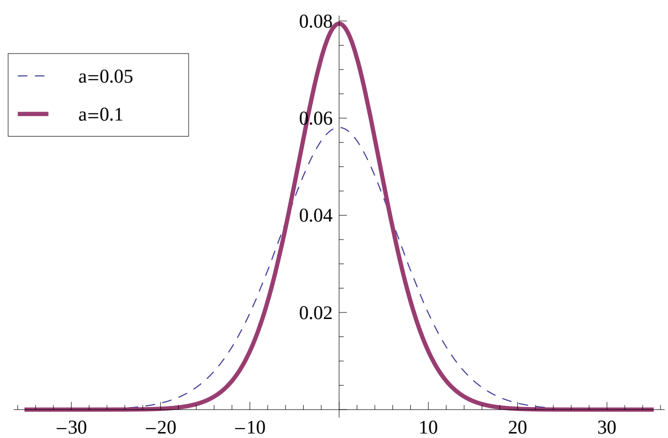

Fig. 2. The plot depicts the stationary solutions (distributions) for the MRW given by equation 15 and the derived stationary solution is in equation 22. By looking these graphs, we can see that as a increases, the stationary distribution has higher peak and thinner tails. Here, for these two stationary solutions, we assumed $\kappa = 1$, $\sigma = 1$.

5 Analytical Time Dependent Solution

To simplify and drop one of parameters in SDE 16, let $Y_t = aX_t$ and apply Ito's lemma to derive:

$$dY_t = \frac{-\alpha Y_t}{2+ \mid Y_t \mid} dt + \sigma' \, dW_t, \tag{24}$$

where $\alpha = a\kappa$ and $\sigma^{'} = a\sigma$. Equivalently, its corresponding PDE is:

$$\frac{\partial P(x\,,\,t)}{\partial t} \;=\; -\frac{\partial}{\partial x}\left(\frac{-\alpha x}{2+\mid x\mid}P(x\,,\,t)\right) \;+\; \frac{D}{2}\,\frac{\partial^2 P(x\,,\,t)}{\partial x^2}, \tag{25}$$

where $D = \sigma^{'}$.

5.1 Analytical Solution for a Special Case

We apply the transformation in equation 26 to equation 25 to pave the way for deriving the time dependent solution as follows:

$$P(x\,,\,t) \;=\; \exp\left(\frac{-2}{D}\,\frac{V(x)}{2}\right)\, q(x\,,\,t), \tag{26}$$

where the $V(x)$ is obtained using the stationary solution in equation 22 as follows:

$$V(x) = \alpha(\mid x\mid -2\ln(2+\mid x\mid)). \tag{27}$$

By applying this transformation, the FPE 25 is converted to a Schrödinger type equation as follows:

$$\frac{\partial q(x\,,\,t)}{\partial t} \;=\; -V_s(x)q(x\,,\,t) \;+\; \frac{D}{2}\,\frac{\partial^2 q(x\,,\,t)}{\partial x^2}, \tag{28}$$

where the $V_s(x)$ is:

$$V_s(x) = \left(\frac{\alpha x^2}{2D} - 1\right)\frac{\alpha}{(2+\mid x\mid)^2}. \tag{29}$$

The PDE 28 might be solved using a superposition method (when the eigenvalues are discrete) as follows:

$$q(x\,,\,t) = \sum_{n=0}^{\infty} a_n(0)e^{-\lambda_n t}\psi_n(x), \tag{30}$$

where $\lambda_n \geq 0$ and $\psi_n(x)$ are eigenvalues and eigenfunctions respectively, and can be derived solving the following eigenvalue problem:

$$\frac{D}{2}\,\frac{d^2\psi_n(x)}{dx^2} - V_s(x)\psi_n(x) = -\lambda_n\psi_n(x), \tag{31}$$

Since by appeal to Lemma 21 $\psi_n(x)$ must be symmetric, we assume $x > 0$. Let $z = \frac{x+2}{a}$ where $a > 0$ is a constant; therefore, we have:

$$\frac{d^2\psi_n(z)}{dz^2} - \left\{\left(-\frac{\alpha^2}{D^2} + \frac{2\lambda_n}{D}\right)a^2 + \frac{\frac{4\alpha^2 a}{D^2}}{z} - \frac{\frac{\alpha(4\alpha-2D)}{D^2}}{z^2}\right\}\psi_n(z) = 0, \tag{32}$$

Let $a = \frac{1}{\sqrt{\frac{4\alpha^2}{D^2} - \frac{8\lambda_n}{D}}}$ where $0 \leq \lambda_n < \frac{\alpha^2}{2D}$, $\kappa = \frac{4\alpha^2 a}{D^2}$ and $\mu =\mid \frac{1}{2} - \frac{2\alpha}{D} \mid$. These assumptions will lead the ordinary differential equation (ODE), 32 to the following format:

$$\frac{d^2\psi_n(z)}{dz^2} - \left\{ -\frac{1}{4} + \frac{\kappa}{z} + \frac{\frac{1}{4} - \mu^2}{z^2} \right\} \psi_n(z) = 0, \tag{33}$$

The ODE, 33 is known as *Whittaker's equation* and has nontrivial general solutions as follows:

$$C_1 W_{\kappa,\mu}\left(\frac{x+2}{a}\right) + C_2 M_{\kappa,\mu}\left(\frac{x+2}{a}\right), \tag{34}$$

where $x >= 0$, and $W_{\kappa,\mu}(z)$ and $M_{\kappa,\mu}(z)$ are Whittaker functions.
For arbitrarily $x \in \Re$, it is easy to show that the general solutions take the following form:

$$C_1 W_{\kappa,\mu}\left(\frac{\mid x \mid +2}{a}\right) + C_2 M_{\kappa,\mu}\left(\frac{\mid x \mid +2}{a}\right), \tag{35}$$

In this subsection, we attempt to solve mean-reverting SDE 24 analytically to find the transition density. Unfortunately, it is difficult to explicitly solve this particular stochastic process. As we saw above for the Whittaker function approach, the presence of the absolute value unfortunately makes impossible to obtain the orthogonality relation to go from equation 35 to a series solution allowing general initial condition to be accommodated. We investigate the reasons for this in the next section.

5.2 Application of Symmetry Group Methods

Consider the stochastic process of the form as follows:

$$dX_t = f(X_t)\,dt + \sigma\,dW_t, \tag{36}$$

Craddock and Dooley [2] deploy *Lie group symmetries* to classify the SDEs like equation 36. They consider the Fokker Planck equation for the probability density $P(x,t)$ associated with SDE 36. Certain classes for the function $f(x)$ allow symmetry to be deployed which transforms $P(x,t) = 1$ to the density associated with equation 36. Here, we state their Theorem (without proof) as follows:

Theorem 51 *There exists a point symmetry for the equivalent heat equations of the SDEs in the form of equation 36 taking $P(x,t) = 1$ to the fundamental solution of the heat equation if and only if $f(x)$ fulfills one of the following*

Ricatti equations:

$$\begin{aligned}
f'(x) + \frac{1}{2} f^2(x) &= \frac{A}{x^2} - \frac{B}{2}, \\
f'(x) + \frac{1}{2} f^2(x) &= \frac{x^2}{8} + \frac{C}{x^2}, \\
f'(x) + \frac{1}{2} f^2(x) &= \frac{C}{(x+2)^2}, \\
f'(x) + \frac{1}{2} f^2(x) &= \frac{2}{3} C x, \\
f'(x) + \frac{1}{2} f^2(x) &= \frac{Cx^2}{2} + D, \qquad (37)
\end{aligned}$$

where A, B, C, and $D \in \Re$ and arbitrary.

Unfortunately, the drift term of this new mean-reverting dynamics 24 does not satisfy any of the Ricatti equations 37; therefore, we could not find explicit solution for this SDE.

6 The New Mean-reverting Process versus Vasicek Process

The Vasicek process was initially introduced by Vasicek [9] to model the evolution of short interest rates so as to capture one essential characteristic of such rates, their mean-reversion. Since the Vasicek process is mean-reverting, simple and analytically tractable, this one-factor model is popular and deployed in various models such as spread processes, credit markets and convenience yield. In Vasicek process, the only factor (state variable), X_t is assumed to follow the stochastic differential equation as follows:

$$dX_t = \kappa\,(\mu \,-\, X_t)\; dt + \sigma\, dW_t, \qquad (38)$$

where $\kappa > 0$ is the speed of mean-reversion, μ is the long-run spread mean, σ is the volatility of the process, and dW_t is the increment of a standard Brownian motion.

Since the long-run mean is not necessarily zero, the dynamics of equation 24 can be generalized as follows:

$$dX_t = \frac{-\kappa(X_t - \mu)}{\mid X_t - \mu \mid +2}\, dt + \sigma\, dW_t, \qquad (39)$$

where μ is the long-run mean.

As discussed above, the new process equation 39 and the Vasicek equation 38 both have the mean-reverting property. We need to compare them to see which one is more appropriate in financial modeling. In these two stochastic processes, when the process tends to deviate away from their long-run means, their drift functions impose force to revert back to long-run mean quite differently. Figure 3 depicts their drift functions. By looking these graphs, we can see that in the Vasicek model as the process attempts to deviate from the long-term mean μ, the drift linearly increases the force to revert back the process to mean μ; however, in the MRW model as the process diverges away from its long-run mean, up to certain ranges of x, the amount of attraction to the mean increases in an approximate linear way just as Vasicek. Unlike Vasicek, however, the restoring force becomes constant in the asymptotic limit. This suggests that this new mean-reverting process has more chance to stay away from the long-run mean for a longer time; in other words, it has relatively heavier tails, which is a typical property of financial processes [4].

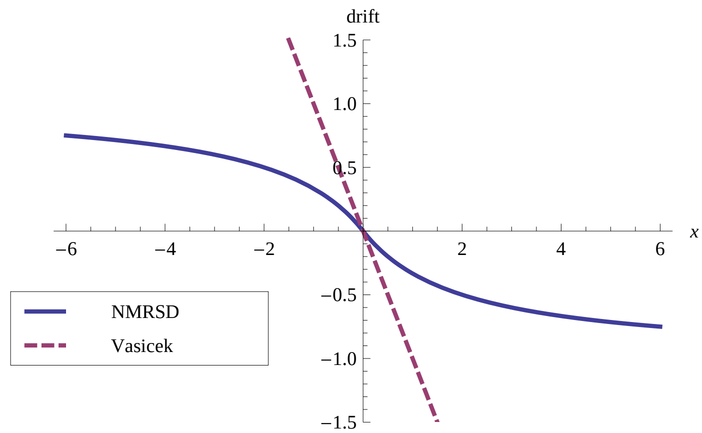

Fig. 3. The plot depicts the drift functions for the new mean-reverting process given in equation 39 and the Vasicek process given in equation 38. We assumed $\kappa = 1$, $\mu = 0$ for both models.

To compare these two processes, we simulate $10,000$ paths for both processes with identical parameters and random values. At each step, we record the empirical variance for all three processes. The constructed variance paths are shown in figure 4. The empirical transition densities also are depicted in figure 5. These figures show that the new mean-reverting dynamics is capable of capturing heavy tails and kurtosis and has more flexibility to capture the reality of the spread processes models compares to that of the Vasicek process.

7 Conclusion

In this paper we introduced a new mean-reverting random walk and derived its SDE limit. Its stationary distribution was derived, but all attempts to

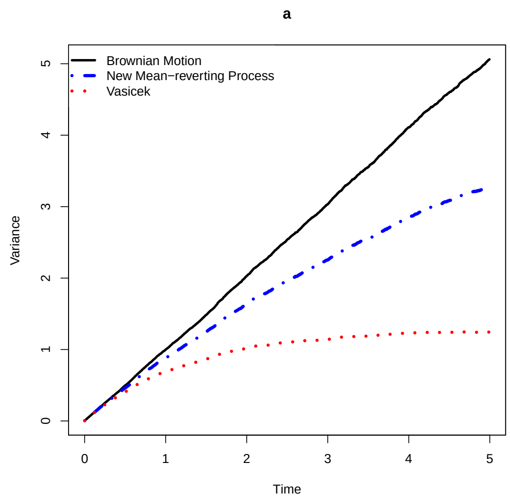

Fig. 4. Time plots of variance and empirical densities for $10,000$ simulated paths. Plot shows how empirical variances evolve through the time in these three models. We assumed $\kappa = 0.4$, $\mu = 0$, $\sigma = 1$, $\Delta t = \frac{1}{252}$, and $T = 5$ for both models by using identical random generated numbers.

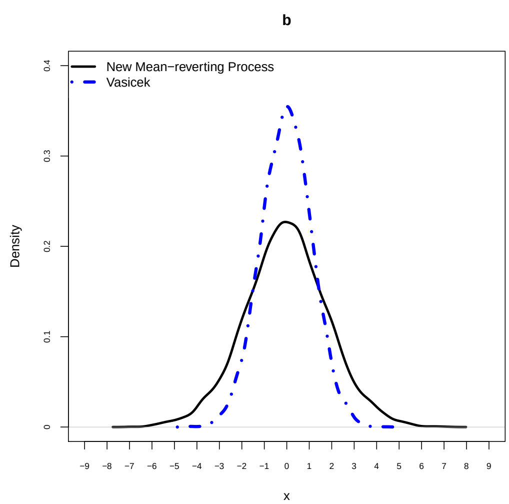

Fig. 5. Time plots of empirical transition densities for $10,000$ simulated paths. Plot shows comparison between empirical densities in these two models after five years. We assumed $\kappa = 0.4$, $\mu = 0$, $\sigma = 1$, $\Delta t = \frac{1}{252}$, and $T = 5$ for both models by using identical random generated numbers.

obtain general solutions for this nonlinear dynamics analytically for transition density failed. We also generalized this new mean-reverting process to equip the diffusion to have long-run mean other than zero. We compared this one-factor stochastic differential equation to Vasicek process [9] and using simulation results, we showed that this new mean-reverting one-factor model has the capability to capture the potential heavy tails of financial processes.

References

1. P. Brémaud. *Markov Chains: Gibbs fields, Monte Carlo simulation, and queues. Vol. 31.* Springer Science & Business Media, 2013.
2. M. J. Craddock and A. H. Dooley. Symmetry group methods for heat kernels. *Journal of Mathematical Physics*, 42(1), 390–418, 2001.
3. M. Davison *Quantitative Finance: A Simulation-based Introduction Using Excel.* CRC Press, 2014.
4. da Cruz, P. Joao and P. G. Lind. Heavy-tails in economic data: fundamental assumptions, modelling and analysis. *arXiv preprint arXiv*:1202.0142, 2012.
5. Oliver C. Ibe. *Elements of Random Walk and Diffusion Processes.* John Wiley & Sons, 2013.
6. N. Kirby. and M. Davison. Using a spark-spread valuation to investigate the impact of corn-gasoline correlation on ethanol plant valuation. *Energy Economics* 32(6), 1221-1227, 2010.
7. H. Risken. *The Fokker-Planck Equation: Methods of Solution and Applications.* Springer Series in Synergetics, 1989.
8. E. Scalas. The application of continuous-time random walks in finance and economics. *Physica A: Statistical Mechanics and its Applications*, 362(2), 225-239, 2006.
9. O. Vasicek. An Equilibrium Characterization of the Term Structure. *Journal of Financial Economics*, 5, 177–188, 1977.
10. G. H. Weiss. Random Walks and Their Applications: Widely used as mathematical models, random walks play an important role in several areas of physics, chemistry, and biology. *American Scientist*, 65-71, 1983.