Correlation Ambiguity

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ABSTRACT

Most papers on ambiguity aversion in the setting of portfolio choice focus on expected return. In this paper, we study portfolio choice under ambiguity aversion to correlation. We show that when ambiguity is large, the optimal portfolio consists of only one risky asset. More generally, the optimal portfolio may contain only a part of all available risky assets. Even though the optimal portfolio is less diversified we show that it is less risky in the sense that it has smaller variance and fewer extreme positions than the standard mean-variance portfolio. With 100 stocks randomly selected from the S&P 500, on average, approximately 20 stocks will be held in the optimal portfolio when sets of ambiguous correlations are given by 95% confidence intervals. Our results suggest that correlation ambiguity has important implications in that it generates fewer extreme portfolio weights and may provide an explanation for under-diversification documented in empirical studies.

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Correlation Ambiguity

In the portfolio choice framework, there are three possible sources of parameter uncertainty: expected return, volatility, and correlation. Because the value function depends on expected return and volatility through the Sharpe ratio, volatility ambiguity is mathematically equivalent to expected return ambiguity. Most previous studies focus on aversion to ambiguous expected return. In this paper, we study correlation ambiguity.

Correlation takes the central role in diversification of portfolio. It is also a major factor when one decides to add a risky asset into an existing portfolio. Low correlation between traditional assets and alternative assets, or emerging market assets is often cited as the reason for including them in portfolios. However, correlations are difficult to estimate accurately, for example, Jagannathan and Ma (2003) note that covariances (and correlations) are imprecisely estimated particularly when the number of assets is large. Thus, it is important to study the effect of correlation ambiguity.

We find that under correlation ambiguity optimal portfolios may contain only one risky asset. We call this anti-diversification[1]. Intuitively, when correlations are totally ambiguous, an optimal portfolio for an ambiguity averse agent should be insensitive to correlations. Only portfolios that consist solely of one asset are insensitive to correlations; hence the optimal strategy for the agent with ambiguity aversion is to hold the asset that has the greatest Sharpe ratio.

When correlations are not completely ambiguous, that is, when correlations can take values in strict subsets of $[-1, 1]$, we have under-diversification in the sense that the optimal portfolio does not contain all risky assets.

We find that the optimal portfolio under correlation ambiguity tends to exclude those assets with large correlations. The rationale is as follows. Investors with ambiguity aversion choose the worst scenario and thus minimize the value function over all feasible correlations.

[1] Goldman (1979) coins this term for holding one risky asset in the optimal portfolio.

Because high correlations lead to high value functions, the optimal portfolio generally does not contain assets with large correlations. As a result, there are fewer extreme positions (portfolio weights of large magnitude) compared to mean-variance portfolios. Extreme positions are considered to be a major reason that causing poor out-of-sample performance of mean-variance portfolios. The aversion to correlation ambiguity can reduce extreme positions, and hence, may improve performance of optimal portfolios under correlation ambiguity.

Anti-diversification or under-diversification implies that optimal portfolios are less diversified and less "balanced". Furthermore, the optimal portfolio has fewer risk free assets than the standard mean-variance portfolio. Thus one may be tempted to conclude that the optimal portfolio is riskier. In fact, we show that the optimal portfolio is less risky in the sense that it has less variance.

In our calibration exercises, the number of risky assets in the optimal portfolio is substantially smaller than the total number of available risky assets. Given 100 randomly-selected U.S. stocks with ambiguous sets being 95% confidence intervals of correlation estimations, the optimal portfolio has approximately 20 stocks. As the degree of correlation ambiguity increases, stocks with a lower Sharpe ratio will tend to be omitted from the optimal portfolio until only the one with the greatest Sharpe ratio remains. In contrast, without ambiguity aversion all 100 stocks are held under the mean-variance framework.

One of the most important insights of modern finance theory is diversification in that the optimal portfolio should contain all available risky assets. Pioneering work of Markowitz (1952), Tobin (1958), and Samuelson (1967) shows that non-positive correlation of asset returns yields an incentive to diversify. MacMinn (1984) further shows that there will still be an incentive to diversify two risky assets with positive correlation to preserve the same mean income and reduce riskiness of portfolio income. This is true for Markowitz's static portfolio choice theory and Merton's dynamic portfolio choice theory. In this paper, we show that under correlation ambiguity there is an incentive to concentrate a portion of available risky assets. In other words, diversification is not optimal in the presence of correlation ambiguity.

Correlation ambiguity and expected return ambiguity have different features. For example, if the expected return ambiguity level is large, no risky asset is held (Boyle, Garlappi, Uppal and Wang (2012)). In contrast, if correlation ambiguity is large, one and only one asset is held.

Uppal and Wang (2003) show that expected return ambiguity can cause biased positions relative to the standard mean-variance portfolio. Guidlin and Liu (2014) examine asset allocation decisions under smooth ambiguity aversion. They find that ambiguity aversion can generate a strong home bias. Izhakian (2012) suggests that holding a fully diversified portfolio is not necessarily optimal in a framework of ambiguity.

As in the literature, we interchange the terms *ambiguity* and *uncertainty*. Both terms are different than *risk*, which has known probability. We refer to Knight (1921), Ellsberg (1961), Maehout (2004) and Hansen and Sargent (2001) for more discussion on ambiguity.

The paper is organized as follows. In Section I, we formulate the framework of portfolio choice with an aversion to correlation ambiguity. In Section II, we study a case of two risky assets. In Section III we study the general case of N risky assets. We disclose conditions under which a risky asset is not included in the optimal portfolio, and investigate features of the optimal portfolio. In Section IV, we study a specific case (anti-diversification). In Section V, we report results regarding empirical calibration. Our conclusions are presented in Section VI. Proofs for certain propositions and a note are in the Appendix.

I. Model Formulation

In this section, we present formulation of portfolio choice under correlation ambiguity.

A. Objective Function

We assume that there is a risk free asset with a constant return r_f, and there are N risky assets with random returns $r_1, ..., r_N$. Let $\mu = (\mu_1, ..., \mu_N)^\top$ denote the expected excess return vector of risky assets, where the convention $^\top$ denotes the transpose, and let Σ denote the variance-covariance matrix of excess returns. Let ϕ_n, $n = 1, ..., N$, denote the dollar amount that is invested in risky asset n and denote the portfolio by $\phi = (\phi_1, ..., \phi_N)^\top$. We consider the following objective

$$\max_{\phi} \mu^\top \phi - \frac{A}{2} \phi^\top \Sigma \phi, \tag{1}$$

where A is the absolute risk aversion coefficient. This objective function can be justified as the utility equivalent of an expected utility with normally distributed returns and a constant absolute risk aversion utility function.

The optimal portfolio without ambiguity of μ and Σ is the solution to the optimization problem (1) given by

$$\phi^* = \frac{1}{A} \Sigma^{-1} \mu.$$

Let σ_n denote the standard deviation of excess return n, $n = 1, ..., N$ and let σ be the diagonal matrix with diagonal entries $\sigma_1, ..., \sigma_N$ in order. Let $\rho = (\rho_{ij})_{1 \le i,j \le N}$ be the correlation matrix of the excess returns, where $\rho_{ij} = 1$ if $i = j$, and ρ_{ij} is the correlation of risky asset i and risky asset j if $i \neq j$. Define $s = (s_1, ..., s_N)^\top$, where $s_n = \mu_n / \sigma_n$ is the Sharpe ratio of risky asset n. Without loss of generality, we assume that Σ is non-singular.

By the change of variable $\psi = \sigma \phi$, the objective problem (1) can be re-written as

$$\max_{\psi} s^\top \psi - \frac{A}{2} \psi^\top \rho \psi. \tag{2}$$

The optimal solution to (2) is $\psi^* = \frac{1}{A}\rho^{-1}s$ and the optimal portfolio ϕ^* is re-represented by

$$\phi^* = \sigma^{-1}\psi^* = \frac{1}{A}\sigma^{-1}\rho^{-1}s.$$

The value function is then obtained by substituting the optimal portfolio into (1) and is given by

$$V = \frac{1}{2A}s^\top\rho^{-1}s.$$

Hence the value function depends on two sets of parameters: ρ and s. While previous studies focus on s (mostly on μ), this paper investigates the role of ρ in portfolio choice when parameters are ambiguous. From the above expressions, volatility ambiguity can be treated the same as expected return ambiguity or Sharpe ratio ambiguity. However, correlation ambiguity is different from Sharpe ratio ambiguity.

As studied in Gilboa and Schmeidler (1989) or Garlappi, Uppal, and Wang (2007), an agent with ambiguity aversion takes the following max-min objective, where minimization reflects the agent's aversion to ambiguity.

$$J = \max_\phi \min_\rho \mu^\top\phi - \frac{A}{2}\phi^\top\Sigma\phi.$$

By applying a version of the minimax theorem (see Theorem 37.3, Rockafellar (1970) and more discussion in Appendix C of this paper), we can change the order of maximization and minimization and obtain the following objective.

$$J = \max_\phi \min_\rho \mu^\top\phi - \frac{A}{2}\phi^\top\Sigma\phi = \min_\rho \max_\phi \mu^\top\phi - \frac{A}{2}\phi^\top\Sigma\phi. \tag{3}$$

Then we can obtain

$$J = \min_{\rho} \frac{1}{2A} s^\top \rho^{-1} s.$$

where $s^\top \rho^{-1} s$ is the sum of squared Sharpe ratios of independent risk, as we will show later.

B. *Ambiguous Set*

We need to specify ambiguous sets in which the minimization in (3) is implemented. It is standard to use confidence intervals as ambiguous sets in the case of expected return ambiguity. We will also use confidence intervals as ambiguity sets of correlations in this paper. However, most of our results do not rely on this specification.

We obtain confidence intervals for point estimations of correlations by a standard method in statistics. Let $R_p = \frac{\sum_{i=1}^n (X_i-\bar{X})(Y_i-\bar{Y})}{\sqrt{\sum_{i=1}^n (X_i-\bar{X})^2(Y_i-\bar{Y})^2}}$ for a paired sample $(X_1, Y_1), ..., (X_n, Y_n)$, with sample mean $(\bar{X}, \bar{Y})$. The Fisher transform $F(R_p) = \frac{1}{2}\ln(\frac{1+R_p}{1-R_p})$ is approximately normally distributed with mean $\frac{1}{2}\ln(\frac{1+p}{1-p})$ and variance $\frac{1}{n-3}$, where p is the population correlation. The confidence bounds are based on the asymptotic normal distribution. These bounds are accurate for large samples when variables have a multivariate normal distribution.

It is well-known that correlations satisfy constraints $|\rho_{ij}| \leq 1$. When $N \geq 3$, there are additional constraints on $\{\rho_{ij}\}_{i<j}$ due to the requirement that ρ must be positive definite. For example, $N = 3$, the three pairs of correlation coefficients must satisfy

$$\rho_{12}^2 + \rho_{13}^2 + \rho_{23}^2 - 2\rho_{12}\rho_{13}\rho_{23} < 1. \tag{4}$$

When confidence intervals are sufficiently small, $\rho_{ij}, 1 \leq i < j \leq N$, inside the confidence intervals should satisfy the constraints. When confidence intervals are large, only those ρ_{ij}'s that satisfy positive definite constraints are chosen from the intervals. Such $\{\rho_{ij}\}$'s will be referred to as *admissible*. One can also specify ambiguous sets of correlations by an elliptical set or a sphere $\sum_{i<j} |\rho_{ij} - \hat{\rho}_{ij}|^2 < \delta$, where $\hat{\rho}_{ij}$ are estimations. Our general results hold true for this setting. For our theoretical results, the optimal ρ satisfies these constraints. When we solve the optimization problem numerically, we use an algorithm under which the positive definite constraint is always binding.

Our formulation is based on a mean-variance static portfolio choice framework. Assuming constant expected returns and variance-covariance matrix, our results also apply to Merton's dynamic portfolio choice framework.

Note that expected return ambiguity and volatility ambiguity can be treated the same as Sharpe ratio ambiguity, because

$$\mu^\top \phi - \frac{A}{2}\phi^\top \Sigma \phi = s^\top \psi - \frac{A}{2}\psi^\top \rho \psi.$$

The expected return ambiguity has been studied extensively in the literature. In an Internet appendix, we derive general results on Sharpe ratio ambiguity that are extensions of existing literature. In the remainder of this paper, we focus on correlation ambiguity, which is different from Sharpe ratio ambiguity.

II. Two Risky Assets

When there are two risky assets ($N = 2$), the optimal portfolio with aversion to correlation ambiguity can be solved in closed form. This case is interesting in itself; it relates to the situation where an investor considers inclusion of a new risky asset in an existing portfolio. The proposition below shows that the agent with ambiguity aversion may hold only one of two risky assets.

PROPOSITION 1: *Assume $s_1 > s_2 \geq 0$. If correlation is completely ambiguous, that is, the ambiguous set of the correlation ρ_{12} is $[-1, 1]$, the agent will only hold asset 1.*

Proof. When $N = 2$,

$$s^\top \rho^{-1} s = s_1^2 + \frac{(s_2 - \rho_{12} s_1)^2}{(1 - \rho_{12}^2)}.$$

It follows that

$$\min_{\rho} \frac{1}{2A} s^\top \rho^{-1} s = \min_{\rho} \frac{1}{2A} \left(s_1^2 + \frac{(s_2 - \rho_{12} s_1)^2}{(1 - \rho_{12}^2)} \right),$$

where the first term in the bracket is the squared Sharpe ratio of asset 1, the second term is the squared Sharpe ratio of a portfolio given by $(-\frac{\sigma_2}{\sigma_1}\rho_{12}, 1)^\top$. Note that asset 1 and the portfolio are uncorrelated.

The first order condition of the above minimization problem leads to $\rho_{12}^* = s_2/s_1 \in [0, 1]$, given that the range of ρ_{12} is $[-1, 1]$, and $s_1 > s_2 \geq 0$. Then the optimal portfolio is

$$\phi^* = \frac{1}{A}\sigma^{-1}(\rho^*)^{-1}s = \frac{1}{A}(\frac{s_1}{\sigma_1}, 0)^\top. \tag{5}$$

Note that the second component is exactly zero, and thus we have anti-diversification. □

When the correlation is completely ambiguous, an ambiguity averse agent will hold a portfolio that is insensitive to correlation.[2] Such portfolios are portfolios with only one risky asset. When there are two risky assets, there are two such portfolios. The one with a higher Sharpe ratio will be chosen.

The above result can be intuitively understood in the following way. Suppose that the risky asset returns are given by

$$r_1 = r_f + \mu_1 + \sigma_1 \epsilon_1,$$

$$r_2 = r_f + \mu_2 + \sigma_2 \epsilon_2,$$

where ϵ_1, ϵ_2 are two sources of shocks with correlation ρ_{12}. For a given correlation ρ_{12}, we

[2]We thank Michael Brennan for pointing this out to us.

can decompose the returns as follows

$$\begin{aligned} r_1 &= r_f + \mu_1 + \sigma_1 \epsilon_1, \\ r_2 &= r_f + \mu_2 + \sigma_2(\rho_{12}\epsilon_1 + \sqrt{1-\rho_{12}^2}\hat{\epsilon}_2), \end{aligned} \tag{6}$$

where $\hat{\epsilon}_2$ is a shock independent of ϵ_1. Both ϵ_1 and $\hat{\epsilon}_2$ are standard normal random variables. Then the return of asset 2 can be expressed as

$$r_2 = r_f + \beta(\mu_1 + \sigma_1\epsilon_1) + \alpha + \sqrt{1-\rho_{12}^2}\sigma_2\hat{\epsilon}_2.$$

where $\beta = \rho_{12}\frac{\sigma_2}{\sigma_1}$ is the regression (in population) coefficient of return 2 on return 1 and $\alpha = (\mu_2 - \rho_{12}\frac{\sigma_2}{\sigma_1}\mu_1)$.

If $\alpha = 0$, there is no compensation for $\hat{\epsilon}_2$ risk, and return 2 is just return 1 plus a pure noise. In this case, $\hat{\epsilon}_2$ is similar to an idiosyncratic risk in the market model with r_1 as the market. If $\alpha \neq 0$, it is the compensation for $\hat{\epsilon}_2$.

The optimal portfolio which solves the optimization problem (1) is given by

$$\begin{aligned} \phi_1^* &= \tfrac{\mu_1}{A\sigma_1^2} - \beta\phi_2^*, \\ \phi_2^* &= \tfrac{\alpha}{A(1-\rho_{12}^2)\sigma_2^2}. \end{aligned} \tag{7}$$

Hence asset 2 will not be held if and only if $\alpha = 0$, which is equivalent to the condition $\rho_{12} = s_2/s_1$. Note that without loss of generality, we assume that the two Sharpe ratios are sorted in descending order $s_1 > s_2 \geq 0$. So $0 \leq s_2/s_1 < 1$.

Given the above optimal portfolio $\phi^* = (\phi_1^*, \phi_2^*)^\top$, the value function in the optimization problem (3) is

$$J = \min_\rho \mu^\top\phi^* - \frac{A}{2}(\phi^*)^\top\Sigma\phi^* = \min_\rho \frac{1}{2A}(\frac{\mu_1^2}{\sigma_1^2} + \frac{\alpha^2}{(1-\rho_{12}^2)\sigma_2^2}) = \min_\rho \frac{1}{2A}(s_1^2 + \frac{(s_2-\rho_{12}s_1)^2}{(1-\rho_{12}^2)}). \tag{8}$$

Therefore, the value function is determined by the sum of squared Sharpe ratios of independent risks. In the above decomposition, the independent risks are ϵ_1 and $\hat{\epsilon}_2$, with Sharpe ratios s_1 and $\sqrt{\alpha^2/(1-\rho_{12}^2)\sigma_2^2} = \sqrt{(s_2 - \rho_{12}s_2)^2/(1-\rho_{12}^2)}$ respectively. Such an uncorrelated/independent decomposition will be exploited again in the case of N risky assets.

Note that the value function is non-monotonic in ρ_{12}. In general, $\rho_{12} = 1$ is not the worst case. In fact, the utility level is unbounded as $\rho_{12} \to 1$ as far as $s_2 < s_1$. When $\rho_{12} = 1$, the two assets are perfectly substitutable as far as risk is concerned, but asset 1 is preferable when risk-return tradeoff is taken into account because it has a higher Sharpe ratio. Thus the agent views this as an arbitrage opportunity and would take infinite positions (infinite long position on asset 1 and infinite short position on asset 2).[3]

Proposition 1 can be extended to the case that the correlation is in a subinterval of $[-1, 1]$, that is, $\rho_{12} \in [\underline{\rho}_{12}, \bar{\rho}_{12}] \subset [-1, 1]$. For this case, we will see a more general portfolio under correlation ambiguity.

Note that

$$\frac{\partial}{\partial \rho_{12}}(s_1^2 + \frac{(s_2 - \rho_{12}s_1)^2}{1-\rho_{12}^2}) = 2s_1^2 \frac{(\frac{s_2}{s_1} - \rho_{12})(-1 + \rho_{12}\frac{s_2}{s_1})}{(1-\rho_{12}^2)^2}.$$

The right hand side is a function of ρ_{12} and its only root in $[-1, 1]$ is s_2/s_1. Thus, the function $(s_1^2 + \frac{(s_2-\rho_{12}s_1)^2}{1-\rho_{12}^2})$ decreases when ρ_{12} is in $[-1, s_2/s_1]$ and increases in $[s_2/s_1, 1]$. From this property one can determine that ρ_{12}^* where

$$\rho_{12}^* = \text{argmin}_{\rho_{12} \in [\underline{\rho}_{12}, \bar{\rho}_{12}]} \frac{1}{2A}\left(s_1^2 + \frac{(s_2 - \rho_{12}s_1)^2}{1-\rho_{12}^2}\right)$$

is the correlation coefficient chosen by the ambiguity averse agent as follows:

If s_2/s_1 is in the range of ambiguity, then $\rho_{12}^* = s_2/s_1$ and only asset 1 will be held. Anti-

[3]If further the condition $s_1 = s_2$ holds along with $\rho_{12} = 1$, the two assets are completely substitutable in both risk and risk-return tradeoff. The optimal portfolio is arbitrary as far as it satisfies $\phi_1^*/\sigma_1 + \phi_2^*/\sigma_2 = s_1/A$.

diversification occurs. This includes the complete ambiguity of Proposition 1 as a special case.

If the range of ρ_{12} is given by $\bar{\rho}_{12} < s_2/s_1$, then $\rho_{12}^* = \bar{\rho}_{12}$ and both assets will be held in long positions .

If the range of ρ_{12} is given by $\underline{\rho}_{12} > s_2/s_1$, then $\rho_{12}^* = \underline{\rho}_{12}$ and both assets will be held. Asset 1 will be held in a long position while asset 2 will be held in a short position.

The above analysis is summarized below as a proposition.

PROPOSITION 2: *Suppose there are only two risky assets. Assume* $s_1 > s_2 \geq 0$, *and* $\rho_{12} \in [\underline{\rho}_{12}, \bar{\rho}_{12}]$ *which is a subinterval of* $[-1, 1]$. *Depending on the values of* $\underline{\rho}_{12}$ *and* $\bar{\rho}_{12}$, *the optimal portfolio under correlation ambiguity is given by*

$$\phi^* = \begin{cases} (\frac{\mu_1}{A\sigma_1^2}, 0)^\top, & \text{if } \underline{\rho}_{12} < \frac{s_2}{s_1} < \bar{\rho}_{12}, \\ \frac{1}{A}\Sigma^{-1}(\bar{\rho}_{12})\mu, & \text{if } \bar{\rho}_{12} < \frac{s_2}{s_1}, \\ \frac{1}{A}\Sigma^{-1}(\underline{\rho}_{12})\mu, & \text{if } \underline{\rho}_{12} > \frac{s_2}{s_1}, \end{cases} \tag{9}$$

where $\Sigma^{-1}(\bar{\rho}_{12})$ *and* $\Sigma^{-1}(\underline{\rho}_{12})$ *are the inverse matrices of* Σ *with* ρ_{12} *replaced by* $\bar{\rho}_{12}$ *and* $\underline{\rho}_{12}$ *respectively.*

We have shown that anti-diversification may occur under correlation ambiguity by Proposition 1 and Proposition 2. We next discuss additional features of the optimal portfolio under correlation ambiguity.

One might expect that aversion to correlation ambiguity leads to an optimal portfolio with more risk free assets than the standard mean-variance portfolio. This is not generally true. The following numerical example illustrates that total allocation of risky assets may increase under ambiguous correlations. Therefore, allocation of risk free assets may be lower under aversion to correlation ambiguity.

Example 1: Suppose $\mu = (0.3, 0.5)^\top, \sigma = diag(0.4, 0.8)$. Then, $s = (0.75, 0.625)^\top$. Assume

$\rho_{12} = 0.82$ and its ambiguous set $[0.6, 0.85]$. Let risk aversion A be 1. Then, the optimal portfolio under ambiguity is $\phi^* = (1.8750, 0)^\top$, while the optimal portfolio without ambiguity is $(1.8124, 0.0382)^\top$.

It follows that the total risky position (1.8506) of the latter portfolio is less than 1.8750, the total risky position of the optimal portfolio under ambiguity. The intuitive reason for this result is that by (7), without ambiguity we can adjust β (or ρ) to make the two risky positions small, while the positions under ambiguity really depend on the ambiguity level, and may be large.

We next examine the relative absolute weight $\frac{\|\phi_1^*\|}{\|\phi_1^*|+\|\phi_2^*\|}$ of asset 1 in the optimal portfolio. For the anti-diversification case, this weight is 1, greater than the weight of asset 1 in the mean-variance portfolio. We can prove that the relative absolute weight of asset 1 in the optimal portfolio is greater than in the standard mean-variance portfolio for all three cases in Proposition 2.

Hence, in the sense of the relative absolute weight, the optimal portfolio under correlation ambiguity is biased toward asset 1, and the portfolio is less "balanced" compared to the standard mean-variance portfolio.

III. N Risky Assets

In this section, we study the general case of N risky assets. We show that the optimal portfolio with aversion to correlation ambiguity typically does not contain all available risky assets. We also show that those assets highly correlated with others tend to not be held in the optimal portfolio. As a result, extreme weights in the optimal portfolio are much fewer than in the standard mean-variance portfolio.

We first study conditions under which an asset is not held. We use asset N as an example.

We write the variance-covariance matrix in a form of blocks as follows.

$$\Sigma = \begin{pmatrix} \tilde{\Sigma}^{\perp} & \sigma_N^2 \tilde{\beta} \\ \sigma_N^2 \tilde{\beta}^{\top} & \sigma_N^2 \end{pmatrix},$$

where $\tilde{\Sigma}^{\perp}$ is the $(N-1) \times (N-1)$ variance-covariance matrix of asset 1,..., asset $N-1$, and

$$\tilde{\beta} = (\rho_{1N} \frac{\sigma_1}{\sigma_N}, ..., \rho_{N-1,N} \frac{\sigma_{N-1}}{\sigma_N})^{\top}.$$

Note that $\tilde{\beta}$ is the population regression coefficient of r_N on $(r_1, ..., r_{N-1})$.

Define

$$\tilde{\alpha}_N = \mu_N - \sigma_N^2 \tilde{\mu}^{\top} (\tilde{\Sigma}^{\perp})^{-1} \tilde{\beta}, \quad \tilde{\sigma}_N^2 = \sigma_N^2 - \sigma_N^4 \tilde{\beta}^{\top} (\tilde{\Sigma}^{\perp})^{-1} \tilde{\beta}, \tag{10}$$

where $\tilde{\mu} = (\mu_1, ..., \mu_{N-1})^{\top}$. Note that $\tilde{\alpha}_N$, or $\tilde{\sigma}_N$ is treated as a function of $\tilde{\Sigma}^{\perp}$ and $\tilde{\beta}$, therefore a function of $\{\rho_{ij}\}_{1 \leq i < j < N}$ and $\{\rho_{iN}\}_{1 \leq i < N}$. Note that

$$s^{\top} \rho^{-1} s = \frac{\tilde{\alpha}_N^2}{\tilde{\sigma}_N^2} + \tilde{\mu}^{\top} (\tilde{\Sigma}^{\perp})^{-1} \tilde{\mu},$$

that corresponds to (8) in the case of two risky assets. This decomposition singles out $\rho_{iN \, 1 \leq i < N}$ in the minimization of the value function and we may obtain the following proposition presenting a necessary condition and a sufficient condition for not holding asset N.

PROPOSITION 3: *If asset N is not held in the optimal portfolio under correlation ambiguity then* $\min_{\rho_{iN,1 \leq i < N}} \tilde{\alpha}_N^2 = 0$. *Conversely, if for all $\rho_{ij}, 1 \leq i < j < N$,* $\min_{\rho_{iN,1 \leq i < N}} \tilde{\alpha}_N^2 = 0$, *then asset N is not held.*

We show the proof in the appendix.

We provide two simple examples to illustrate the above proposition. One example is the case of $N = 2$. In this case $\tilde{\alpha}_2 = \mu_2 - \frac{\sigma_2}{\sigma_1} \rho_{12} \mu_1$, $\tilde{\sigma}_2^2 = \sigma_2^2 (1 - \rho_{12}^2)$, and $\tilde{\Sigma}^{\perp}$ is the constant σ_1^2.

The condition $\min_{\rho_{12}} \tilde{\alpha}_2^2 = \min_{\rho_{12}}(\mu_2 - \rho_{12}\sigma_2/\sigma_1)^2 = 0$ becomes a sufficient and necessary condition for not holding asset 2. The condition leads to Proposition 2 of the case of two risky assets.

Another example is $\rho_{iN}^* = 0, 1 \le i < N$. In this case, the condition $\min_{\rho_{iN}} \tilde{\alpha}_N^2 = 0$ is equivalent to $\mu_N = 0$. Hence asset N which is uncorrelated with all other assets in the worst case is not held if and only if $\mu_N = 0$.

Proposition 3 provides a necessary and a sufficient condition for not holding an asset. As an application of the proposition, we argue that risky assets in the optimal portfolio have small correlations.

First, large correlations generate a high value function that is likely to be ruled out by the minimization nature of ambiguity aversion. Thus, optimal correlations tend to be low. Second, if the correlation of a pair of risky assets is close to 1 and the assets have very close Sharpe ratios, the condition $\tilde{\alpha}_N^2 = 0$ is more likely to be satisfied in the ambiguous set of correlation (close 1). By Proposition 2 of the two assets case, one of the pair will not be held. So risky assets in the optimal portfolio under correlation ambiguity tend to have small correlations. As a consequence, extreme positions in the optimal portfolio may not occur as often as in the standard mean-variance portfolio. We will discuss implications of this result in a calibration exercise later.

Note that $\min_{\rho_{iN}} \tilde{\alpha}_N^2 = 0$ is not easy to check. Instead, the following result is useful in empirical calibration exercises.

COROLLARY 1: *The optimal portfolio under correlation ambiguity is given by*

$$\phi^* = \frac{1}{A}\sigma^{-1}(\rho^*)^{-1}s,$$

where ρ^* *is given by*

$$\rho^* = \arg\min_{\rho} s^\top \rho^{-1} s. \tag{11}$$

Corollary 1 is actually from (3). We list it as a corollary here because it provides us a quick way to find the optimal portfolio from a large set of risky assets under correlation ambiguity. Given s, the optimization problem (11) can be transformed to a semi-definite programming (SDP) problem. This is used in our calibration exercise. It's worth mentioning that the positive definite constraint is binding when we solve the problem numerically. We show detailed transforming steps in the appendix.

The following proposition suggests that under-diversification is associated with interior solutions to the minimization problem. It also suggests that under-diversified portfolios may occur quite generally under correlation ambiguity.

PROPOSITION 4: *If* $\rho^*_{ij} \in (\underline{\rho}_{ij}, \bar{\rho}_{ij})$*, then either* $\phi^*_i = 0$ *or* $\phi^*_j = 0$ *for* $1 \le i \ne j \le N$*. In other words, if an optimal correlation is achieved at an interior point of the ambiguity set, then at least one of the two corresponding risky assets will not be held.*

We show the proof in the appendix. The intuition behind the proposition is that the value function depends on ρ_{ij} through $\rho_{ij}\phi_i\phi_j$. Hence if changing ρ_{ij} does not affect the utility then $\phi^*_i\phi^*_j$ must be zero, because this product is the coefficient of ρ^*_{ij} in the optimal utility function.

In our empirical calibration exercise, the optimal portfolio under correlation ambiguity is under-diversified even when we replace confidence intervals by a sphere or an ellipsoidal set as the ambiguity set of correlations. In contrast, under expected return ambiguity, there is no under-diversification for an ellipsoidal or a sphere ambiguity set. This can be seen from the case of two risky assets discussed in the Internet Appendix of this paper, or from Proposition 2, Garlappi, Uppal, and Wang (2007). In fact, under-diversification (holding only part of risky assets) shown in Boyle, Garlappi, Uppal and Wang (2012) may not occur if the ellipsoidal ambiguous set $\{\mu : (\mu - \hat{\mu})^\top \Sigma^{-1}(\mu - \hat{\mu}) < \delta\}$ is used as the range of expected returns there.

IV. Anti-Diversification

In this section, we show that, correlations with sufficient ambiguity lead to anti-diversification in the sense that the optimal portfolio consists of exactly one risky asset, even though there are $N > 1$ risky assets available. A sufficient and necessary condition for occurrence of anti-diversification is presented.

We index the risky asset with the greatest Sharpe ratio as asset 1. The optimal portfolio with aversion to correlation ambiguity should produce a value function not less than the portfolio of investing only one of the risky assets. If only investing asset i, the optimal portfolio is $\frac{1}{A}\frac{\mu_i}{\sigma_i^2}$ and the value function is $\frac{1}{2A}s_i^2$. So the optimal value function J satisfies $J \geq \frac{1}{2A}s_i^2, i = 1, 2, ..., N$. Thus, we have $J \geq \frac{1}{2A}\max_i s_i^2 = \frac{1}{2A}s_1^2$. That is, investing in asset 1 reaches the lower bound of the value function and holding one of other assets can not be optimal. We write this result and its consequence as a proposition below.

PROPOSITION 5: *Suppose $s_1 > \max\{s_2, ..., s_N\} \geq 0$. Then the value function is greater than or equal to $\frac{1}{2A}s_1^2$. When anti-diversification occurs, only the risky asset with the greatest Sharpe ratio among all available risky assets is held.*

A direct consequence of the above proposition and Proposition 4 is anti-diversification: If all correlations are completely ambiguous ($\underline{\rho}_{ij} = -1, \bar{\rho}_{ij} = 1$ for all $i, j = 1, 2, ..., N$), then at most one ϕ_i^* is not zero, hence only one asset will be held and anti-diversification occurs. By Proposition 5, the risky asset with the greatest Sharpe ratio is held.

To study the N risky assets case, let us consider the following change of variables.

$$\begin{aligned} \varphi_1 &= \phi_1 + \beta_2\phi_2 + ... + \beta_N\phi_N, \\ \varphi_i &= \phi_i, \ i = 2, 3, ..., N. \end{aligned}$$

In matrix notation, we can write

$$\varphi = \begin{pmatrix} 1 & \beta^\top \\ 0 & I_{N-1} \end{pmatrix} \phi, \tag{12}$$

with $\varphi = (\varphi_1, ..., \varphi_N)^\top, \phi = (\phi_1, ..., \phi_N)^\top$, $\beta = (\beta_2, ..., \beta_N)^\top$, and for each $i = 2, ..., N$,

$$\beta_i = \frac{\rho_{1i}\sigma_i}{\sigma_1}$$

is the beta coefficient of regression (in population) coefficient of r_i on r_1.

We can express Σ matrix in the following block-diagonal form

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_1^2\beta^\top \\ \beta\sigma_1^2 & \Sigma^\perp \end{pmatrix}$$

where $\Sigma^\perp$ is the $(N-1) \times (N-1)$ variance-covariance matrix for assets 2, . . ., N. Note that

$$\begin{pmatrix} 1 & 0 \\ -\beta & I_{N-1} \end{pmatrix} \begin{pmatrix} \sigma_1^2 & \beta^\top\sigma_1^2 \\ \beta\sigma_1^2 & \Sigma^\perp \end{pmatrix} \begin{pmatrix} 1 & -\beta^\top \\ 0 & I_{N-1} \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \Sigma^\perp - \sigma_1^2\beta\beta^\top \end{pmatrix}. \tag{13}$$

We then have

$$\phi^\top\Sigma\phi = \varphi_1^2\sigma_1^2 + \hat{\varphi}^\top(\Sigma^\perp - \sigma_1^2\beta\beta^\top)\hat{\varphi},$$

where $\hat{\varphi} = (\varphi_2, ..., \varphi_N)^\top$, and

$$\mu^\top \phi = (\mu_1, \hat{\mu}^\top - \beta^\top \mu_1)\varphi,$$

where $\hat{\mu} = (\mu_2, ..., \mu_N)^\top$. Thus, the agent's objective function without ambiguity becomes

$$\max_{\phi} \mu^\top \phi - \frac{A}{2}\phi^\top \Sigma \phi = \max_{\varphi} \mu_1 \varphi_1 - \frac{A}{2}\sigma_1^2 \varphi_1^2 + (\hat{\mu} - \beta\mu_1)^\top \hat{\varphi} - \frac{A}{2}\hat{\varphi}^\top (\Sigma^\perp - \sigma_1^2 \beta\beta^\top)\hat{\varphi},$$

and the maximizer can be separated into two components as follow

$$\varphi_1^* = \frac{\mu_1}{A\sigma_1^2} \quad \text{and} \quad \hat{\varphi}^* = \frac{1}{A}(\Sigma^\perp - \sigma_1^2 \beta\beta^\top)^{-1}\alpha, \tag{14}$$

where

$$\alpha = \hat{\mu} - \beta\mu_1.$$

Note that we use the same notations of α and β as in Section II because they are identical when $N = 2$.

Next, by the relation (12), we have

$$\phi_1^* + \sum_{n=2}^{N} \beta_n \phi_n^* = \frac{\mu_1}{A\sigma_1^2}$$

and

$$(\phi_2^*, ..., \phi_N^*)^\top = \frac{1}{A}(\Sigma^\perp - \sigma_1^2 \beta\beta^\top)^{-1}\alpha.$$

Therefore,

$$\max_{\phi} \mu^\top \phi - \frac{A}{2}\phi^\top \Sigma \phi = \frac{1}{2A}(\frac{\mu_1^2}{\sigma_1^2} + \alpha^\top (\Sigma^\perp - \sigma_1^2 \beta\beta^\top)^{-1}\alpha).$$

Then, the objective function with aversion to correlation ambiguity becomes

$$\min_{\rho} \frac{1}{2A}\left(\frac{\mu_1^2}{\sigma_1^2} + \alpha^\top (\Sigma^\perp - \sigma_1^2 \beta\beta^\top)^{-1}\alpha\right), \tag{15}$$

which is the N risky assets version of (8).

From (15), because $(\Sigma^{\perp} - \sigma_1^2 \beta\beta^{\top})^{-1}$ is positive definite, $\frac{1}{2A}\frac{\mu_1^2}{\sigma_1^2}$ is the minimum if and only if $\alpha = 0$. In this case, the optimal portfolio has only one risky asset which is asset 1 and we have anti-diversification.

Note that $\alpha = 0$ is equivalent to $\rho_{1i}^* = \frac{s_i}{s_1}, i = 2, ..., N$. This is just the N-dimensional extension of the two-risky-asset case we study in Section II. The intuition for this condition is the same as given for the case of $N = 2$.

We also note that $\rho_{1i}, i = 2, ..., N$, can be independently specified, as long as $|\rho_{1i}| < 1$. [4] However, after $\rho_{1i}, i = 2, ..., N$, are given, $\{\rho_{ij}\}_{2 \le i < j}$ can not be specified independently. They have to satisfy additional constraints for ρ to be positive definite. For example, when $N = 3$, ρ_{12} and ρ_{13} can be independently specified to take any value between -1 and 1. But given ρ_{12} and ρ_{13}, we can no longer specify ρ_{23} to take any value between -1 and 1, due to the constraint (4).

The above analysis yields that the condition $\alpha = 0$ is a sufficient and necessary condition for occurrence of anti-diversification if $\rho_{1i}^* = s_i/s_1$ is admissible.

PROPOSITION 6: *Anti-diversification occurs if and only if* $\rho_{1i}^* = s_i/s_1, i = 2, ..., N$, *is admissible.*

An alternative way to understand this proposition is as follows. The decomposition (15) implies that we can construct two groups of assets: one denoted by X consisting of asset 1 only, another group denoted by Y with return vector

$$(-\beta, I_{N-1})(r_1, r_2, ..., r_N)^{\top} = (r_2 - \beta_2 r_1, ..., r_N - \beta_N r_1)^{\top}.$$

[4]This is because for any values of $\rho_{1i}, i = 2, ..., N$, we can find values between -1 and 1 for other ρ_{ij}, such that ρ is positive definite. For example, we can let $\rho_{ij} = \rho_{1i}\rho_{1j}$, for $1 < i \neq j \le N$, then the correlation matrix ρ is positive definite for any values of ρ_{1i}, ρ_{1j} between -1 and 1.

Then, Y has expected return vector

$$(-\beta, I_{N-1})\mu = \hat{\mu} - \beta\mu_1 = \alpha$$

and variance-covariance matrix

$$(-\beta, I_{N-1})^\top \Sigma(-\beta, I_{N-1}) = \Sigma^\perp - \sigma_1^2 \beta\beta^\top.$$

One can check that X is uncorrelated with each member of Y, and hence uncorrected with any portfolio over Y. When correlations are ambiguous and the agent is averse to ambiguity, the agent will consider the worst case in which the second term of (15) is minimized over the ambiguous sets of correlations. The value function is the sum of the squared Sharpe ratio of X and the optimal Sharpe ratio of Y divided by $2A$. Only the latter Sharpe ratio contains ρ; in the worst scenario ($\alpha = 0$) this Sharpe ratio is zero and asset $2, ..., N$, are not held. Note that α depends on ρ_{1n} only, not on all ρ_{ij}'s. Hence, it induces anti-diversification that $\alpha = 0$, given $\rho_{1i}^* = s_i/s_1, i = 2, ..., N$, is admissible.

When anti-diversification occurs, only asset 1 with the greatest Sharpe ratio is held. Any portfolio formed with the rest of the risky assets should have a lower Sharpe ratio than asset 1. Here is an example for the three risky assets case. When anti-diversification occurs, by Proposition 6, it holds that $\rho_{12}^* = s_2/s_1, \rho_{13}^* = s_3/s_1$, and the correlation matrix shall be positive definite. Substituting $\rho_{12} = \rho_{12}^*, \rho_{13} = \rho_{13}^*$ into (4), it follows that

$$s_2^2 + s_3^2 - 2\rho_{23}s_2s_3 < (1 - \rho_{23}^2)s_1^2. \tag{16}$$

On the other hand, the Sharpe ratio of the optimal portfolio formed by using asset 2 and 3 is

$$(s_2, s_3)\begin{pmatrix} 1 & \rho_{23} \\ \rho_{23} & 1 \end{pmatrix}^{-1}\begin{pmatrix} s_2 \\ s_3 \end{pmatrix} = \frac{s_2^2 + s_3^2 - 2\rho_{23}s_2s_3}{1-\rho_{23}^2}.$$

This is smaller than s_1 if and only if

$$s_2^2 + s_3^2 - 2\rho_{23}s_2s_3 < (1-\rho_{23}^2)s_1^2,$$

which is the same as (16). So any portfolio of asset 2 and asset 3 has a lower Sharpe ratio than asset 1 if anti-diversification occurs. However, the converse is not true. One may consider a case without correlation ambiguity.

When $\rho_{1i}^* = s_i/s_1, i = 2, ..., N$, and $\rho_{ij}^* = s_is_j/s_1^2$, $1 < i < j \leq N$, the correlation matrix $\rho^* = (\rho_{ij}^*)$ is positive definite. This gives us a simple sufficient condition for occurrence of anti-diversification by Proposition 6.

PROPOSITION 7: *If s_i/s_1 is contained in the ambiguous sets of ρ_{1i} for $i = 2, ..., N$, and s_is_j/s_1^2 is contained in the ambiguous sets of ρ_{ij} for $1 < i < j \leq N$, anti-diversification occurs.*

The sufficient condition in Proposition 7 is automatically satisfied when correlations are completely ambiguous, that is, ambiguous sets are $[-1, 1]$. We obtain a general result for the case of N risky assets, in accordance with Proposition 1, where only two risky assets are assumed.

COROLLARY 2: *Assume $s_1 > \max\{s_2, s_3, ..., s_N\} \geq 0$. If correlations are completely ambiguous, that is, the ambiguous set for each correlation coefficient ρ_{ij} is $[-1, 1]$, then the agent will hold asset 1 only.*

When anti-diversification occurs, only the asset with the greatest Sharpe ratio will be held. However, when anti-diversification does not occur, this asset may not be held in the optimal portfolio while the one with the smallest Sharpe ratio may be held, depending on correlations and ambiguous levels. Other assets may combine to achieve a higher Sharpe ratio than asset 1.

The following numerical example illustrates the fact.

Example 2: Suppose $\mu = (0.3, 0.2, 0.1)^\top, \sigma = diag(0.4, 0.3, 0.2)$. Then, $s = (0.75, 0.667, 0.5)^\top$. Suppose further that correlation ambiguity levels are $\rho_{12} \in [0.6, 0.9], \rho_{13} \in [0.5, 0.7], \rho_{23} = 0$. Then, we find the optimal portfolio under correlation ambiguity is $\phi^* = (0, 2.22, 2.50)^\top$, and the value function is 0.3472, given risk aversion coefficient $A = 1$.

In this example, correlation between asset 2 and asset 3 contains no ambiguity, and their optimal combination can reach a Sharpe ratio $\sqrt{s_2^2 + s_3^2} = 0.8334 > 0.75 = s_1$. Asset 1 is dropped due to its ambiguity in correlations with asset 2 and 3, even though it has the greatest Sharpe ration among the three assets.

The example also suggests that an asset with the smallest Sharpe ratio may be held in the optimal portfolio.

Under the mean-variance expected utility, the portfolio is given by $\phi^* = \frac{1}{A}\Sigma^{-1}\mu$. The probability of $\phi_i^* = 0$ for some i is zero. Alternatively, if Σ is non-singular, then each asset i has an independent risk and a risk premium α_i associated with the risk. Thus, under the expected utility, all available risky assets will be held in general. In contrast, under aversion to correlation ambiguity, minimizing over ρ singles out $\alpha_i = 0$. In this case, $\alpha_i = 0$ and anti-diversification occurs for sure.

It seems that the optimal portfolio containing only one risky asset is riskier than the standard mean-variance portfolio which contains all risky assets. For example, suppose $s_1 > s_i, i = 2, ..., N$, $\frac{\mu_1}{\sigma_1^2} = ... = \frac{\mu_N}{\sigma_N^2}$, and the optimal portfolio under correlation ambiguity is given by $\phi^* = \frac{\mu_1}{A\sigma_1^2}(1, 0, ..., 0)^\top$. Note that under the expected utility, the optimal portfolio is given by $\phi_{MV}^* = \frac{1}{A}(\frac{\mu_1}{\sigma_1^2}, ..., \frac{\mu_N}{\sigma_N^2})^\top = \frac{\mu_1}{A\sigma_1^2}(1, 1, ..., 1)^\top$, assuming $\rho = I_N$. A portfolio with $\phi^* = \frac{1}{A}\frac{\mu_1}{\sigma_1^2}(1, 0, ..., 0)$ seems to be much more "imbalanced" thus riskier than a portfolio with

$\phi^*_{MV} = \frac{1}{A}\frac{\mu_1}{\sigma_1^2}(1, ..., 1)^\top$. However, the variance of the portfolio ϕ^* is

$$(\phi^*)^\top \Sigma \phi^* = \frac{1}{A^2} s_1^2$$

while the variance of the mean-variance portfolio ϕ^*_{MV} is

$$(\phi^*_{MV})^\top \Sigma \phi^*_{MV} = \frac{1}{A^2} \sum_i s_i^2 > \frac{1}{A^2} s_1^2.$$

Thus, the portfolio ϕ^* actually has a lower variance and thus is less risky. In this example, we assume $\rho = I_N$ to get the mean-variance portfolio. In fact, for any admissible ρ, the same conclusion holds. Note $\phi^* = \frac{1}{A}\sigma^{-1}\rho^{-1}s$ and

$$\min_\rho \max_\phi \phi^\top \mu - \frac{A}{2}\phi^\top \Sigma \phi = \frac{1}{2A} \min_\rho s^\top \rho^{-1} s = \frac{A}{2} \min_\rho (\phi^*)^\top \sigma\rho\sigma\phi^*.$$

The term $(\phi^*)^\top \sigma\rho\sigma\phi^*$ is in deed the variance of the return regarding portfolio ϕ^*. Hence the optimal portfolio under the ambiguity has the minimum variance among all portfolios in the form $\frac{1}{c}\sigma^{-1}\rho^{-1}s$ for any non-zero constant scale c. We summarize this result as a proposition.

PROPOSITION 8: *The optimal portfolio under correlation ambiguity has smaller variance than the standard mean-variance portfolio.*

The proposition says that the optimal portfolio is less risky than the standard mean-variance portfolio.

Goldman (1979) shows that in an infinite time horizon, buy-and-hold strategy will result in anti-diversification. In his paper, the asset with the highest risk and risk aversion adjusted expected return will be held. In our paper, it is the asset with the highest Sharpe ratio.

V. Empirical Calibration

We calibrate our model using U.S. stock market data and study the optimal portfolio under correlation ambiguity. We use monthly data of S&P 500 adjusted for dividends. The data set spans from January, 1993 to December, 2012, for a total of 240 months. After filtering out incomplete data, there are 319 stocks that remain for study. We compute mean excess return, variance and correlation of excess returns for these stocks, using average monthly LIBOR as the risk free return.

Brief statistical information regarding the data are shown in Table 1. Of a total of 50721 ($319 \times 318/2$) estimated correlations, 0.19% have a p-value greater than 5%; hence, most of the correlations are significant. The maximum correlation is 0.8365, and the minimum is -0.6385. Neither is close to the value -1 or 1. The average length of 95% confidence interval is 0.2357. The length rises to 0.3088 at the 99% confidence interval. A higher confidence interval level corresponds to a higher level of ambiguity. As is standard in the literature, confidence intervals are used as ambiguous sets of correlations.

In preceding sections, we show that aversion to correlation ambiguity can generate under-diversification. To quantitatively study the phenomenon of under-diversification, we randomly choose a sample group of stocks from the S&P 500 for $N = 10, 20, ..., 100$, and then compute the optimal portfolio under ambiguous correlations for investors with ambiguity aversion. Ambiguous intervals of estimated correlations result from different levels of confidence subject to the positive definite constraint. We repeat this procedure 100 times, and obtain 100 optimal portfolios. The average number of stocks in the optimal portfolios for each N is the typical size of the investors' optimal portfolios when they encounter N available stocks.

The result is reported in Figure 1. When there are 100 stocks, optimal portfolios consist of 22 stocks, on average, given the 95% confidence intervals as the ambiguous sets of the correlations. Optimal portfolios consist of approximately 18 and 24 stocks when investors select from the 100 stocks and use 90% and 99% confidence intervals, respectively, as the ambiguity sets of the correlations. Hence, the aversion to correlation ambiguity generates under-diversification, as documented in the empirical studies.

We let risk aversion $A = 1$ in our empirical tests. It is important to note that risk aversion solely affects the magnitude of risky positions by scaling; it does not change the choices of risky assets in the optimal portfolio under correlation ambiguity. In fact, we may obtain a corollary from Proposition 4 as follows.

COROLLARY 3: *The set of risky assets in the optimal portfolio under correlation ambiguity is independent of the risk aversion.*

Proof. By Proposition 4, we observe that whether asset i has zero positions depends on whether ρ_{ij}^* falls into the ambiguous set; this, in addition to the objective function in (11), is independent of risk aversion A. $\square$

Empirical studies document that investors usually hold much less risky assets than they could. For example, Campbell (2006) finds that the financial portfolios of households contain only a few risky assets. Goetzmann and Kumar (2008) report that the majority of individual investors hold a single digit number of assets in a sample data set from 1991-1996. Among many empirical findings regarding under-diversification from various data sets, we refer to Mitton and Vorkink (2007), Calvet, Campbell, and Sodini (2008), and Ivković, Sialm, and Weisbenner (2008). Our result suggests that correlation ambiguity may be an explanation for their findings.[5]

Next, we focus on one randomly selected sample of 100 stocks. Results are reported in Table II. When 95% confidence intervals are used as ambiguous sets for correlations, we

[5]There are other explanations for under-diversification. For example, Brennan (1975) finds that the optimal number of risky assets in a portfolio is small when there are fixed transaction costs. Goldman (1979) shows that, for buy-and-hold strategies, an infinite time horizon leads to holding one risky asset. Liu (2014) proposes a model in which under-diversification may be caused by solvency requirements in the presence of committed consumption. Roche, Tompaidis, and Yang (2013) suggest that financial constraints can lead to under-diversification. Nieuwerburgh and Beldkamp (2005) propose an explanation based on information costs.

obtain an optimal portfolio that consists of 20 stocks. The maximum, minimum and average of the Sharpe ratios of stocks held in the optimal portfolio are 0.1674, -0.0489, and 0.0859, respectively, and the corresponding quantities are 0.1153, -0.0064, and 0.0600, for stocks with zero positions in the optimal portfolio. The former group has a higher average Sharpe ratio than the latter. The distributions of the Sharpe ratios in the optimal portfolio and in the entire sample are presented in Figure 2. Note that the stock with the greatest Sharpe ratio is held, but not all the held stocks have large Sharpe ratios. Point estimations of correlations that determine positions of ambiguity sets, and ambiguity levels that determine size of ambiguity sets, matter here as well as Sharpe ratios.

By Proposition 2, a stock will not be held in the optimal portfolio if the ratio of its Sharpe ratio to the maximal Sharpe ratio falls into the ambiguous interval of the correlation. Given the average correlation of 0.2220, and the maximal Sharpe ratio of 0.1674, we can expect that stocks with a Sharpe ratio of approximately $0.0372 (= 0.2220 \times 0.1674)$ tend not to be held. This analysis is confirmed by Figure 2, which shows that stocks with Sharpe ratios near the point 0.03 are not held.

When the level of ambiguity is greater, the average Sharpe ratio in the optimal portfolio is higher. This finding suggests that stocks with low Sharpe ratios are more likely to be eliminated from the optimal portfolio as the ambiguity level increases, until the one with the greatest Sharpe ratio remains.

In Figure 3, we compare the optimal portfolio with the standard mean-variance portfolio constructed from sample moments. Table III lists the exact values of non-zero positions in the optimal portfolio and corresponding positions in the mean-variance portfolio. The extreme weights in the mean-variance portfolio are significantly reduced in the optimal portfolio under ambiguity. For example, the largest weight 6.1355 in the mean-variance portfolio is reduced to 0.1604 in the optimal portfolio, and the smallest -2.5868 is reduced to -0.0483.

The reason that the extreme allocations are largely reduced in the optimal portfolio under the ambiguity can be inferred from Figure 4. This figure plots distributions of correlations of all 100 stocks, as well as of those stocks in the optimal portfolio. As argued after Proposition 3, many stocks with large correlations are eliminated from the optimal portfolio. Stocks in the optimal portfolio tend to have small correlations, and extreme weights are less likely when the correlations are closer to zero. Thus, the optimal portfolio under correlation ambiguity likely has fewer extreme positions than the mean-variance portfolio.

Extreme positions are considered to be a major reason that causes poor out-of-sample performance.[6] The aversion to correlation ambiguity can reduce extreme positions, and hence, may improve performance of optimal portfolios under correlation ambiguity. In fact, in our out-of-sample tests for various data sets not reported in this paper, the aversion to correlation ambiguity generates portfolios with a more stable and higher Sharpe ratio than the mean-variance portfolios. These findings regarding performance are similar to Garlappi, Uppal, and Wang (2007). They study portfolio choice with aversion to expected return ambiguity. By examining extreme positions, we provide one possible reason why performance of optimal portfolios under correlation ambiguity can be better than mean-variance portfolios. This observation relates to the empirical finding that concentrated holdings are associated with superior performance of mutual funds (Kacperczyk, Sialm, and Zheng (2005)).

VI. Conclusions

In this paper, we study optimal portfolios for an agent who is averse to correlation ambiguity. We prove that anti-diversification (the optimal portfolio has only one risky asset) occurs when correlations are sufficiently ambiguous. Generally, we have under-diversification in the sense that optimal portfolios contain only part of available risky assets. Using plausible levels of ambiguity, an optimal portfolio of 20 stocks is created from 100 randomly-selected stocks from the S&P 500. Correlation ambiguity may provide an explanation for under-diversification documented in the literature.

[6]For example, Jagannathan and Ma (2003) and DeMiguel, Garlappi, Nogales, and Uppal (2009) show out-of-sample performance can be improved by constraining the weights(hence, reducing extreme positions) in optimal portfolios.

Under correlation ambiguity the optimal portfolio with anti-diversification or under-diversification is less diversified and less balanced in the sense that many available assets are excluded. Also, the optimal portfolio may contain fewer risk free assets than the standard mean-variance portfolio. Thus one may be tempted to conclude that the optimal portfolio is riskier. In fact, the optimal portfolio is less risky because it has less variance. In addition, the optimal portfolio has fewer “extreme” positions because risky assets with high correlations are more likely to be omitted from the optimal portfolio under correlation ambiguity.

Appendix A. Proof of Propositions

Proof of Proposition 3: Let

$$\psi = (\psi_1, ..., \psi_N)^\top = \begin{pmatrix} I_{N-1} & \sigma_N^2(\tilde{\Sigma}^\perp)^{-1}\tilde{\beta} \\ 0 & 1 \end{pmatrix} \phi.$$

and $\tilde{\psi} = (\psi_1, ..., \psi_{N-1})$. Note that

$$\begin{pmatrix} I_{N-1} & 0 \\ -\tilde{\beta}^\top(\tilde{\Sigma}^\perp)^{-1} & 1 \end{pmatrix} \begin{pmatrix} \tilde{\Sigma}^\perp & \sigma_N^2\tilde{\beta} \\ \sigma_N^2\tilde{\beta}^\top & \sigma_N^2 \end{pmatrix} \begin{pmatrix} I_{N-1} & -(\tilde{\Sigma}^\perp)^{-1}\tilde{\beta} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \tilde{\Sigma}^\perp & 0 \\ 0 & \sigma_N^2 - \tilde{\beta}^\top(\tilde{\Sigma}^\perp)^{-1}\tilde{\beta}\sigma_N^4 \end{pmatrix}.$$

We then have

$$\phi^\top\Sigma\phi = \tilde{\psi}^\top\tilde{\Sigma}^\perp\tilde{\psi} + \psi_N^2(\sigma_N^2 - \tilde{\beta}^\top(\tilde{\Sigma}^\perp)^{-1}\tilde{\beta}\sigma_N^4)$$

and

$$\mu^\top\phi = \tilde{\mu}^\top\tilde{\psi} + (\tilde{\mu}^\top\sigma_N^2(\tilde{\Sigma}^\perp)^{-1}\tilde{\beta} + \mu_N)\psi_N.$$

Thus the objective function can be written as follows.

$$\begin{aligned}
\min_\rho \max_\phi (\mu^\top\phi - \frac{A}{2}\phi^\top\Sigma\phi) &= \min_\rho \max_\psi \tilde{\mu}^\top\tilde{\psi} + (-\tilde{\mu}^\top\sigma_N^2(\tilde{\Sigma}^\perp)^{-1}\tilde{\beta} + \mu_N)\psi_N \\
&\quad - \frac{A}{2}\left(\tilde{\psi}^\top\tilde{\Sigma}^\perp\tilde{\psi} + \psi_N^2(\sigma_N^2 - \tilde{\beta}^\top(\tilde{\Sigma}^\perp)^{-1}\tilde{\beta}\sigma_N^4)\right) \\
&= \frac{1}{2A}\min_\rho\left(\frac{\tilde{\alpha}_N^2}{\tilde{\sigma}_N^2} + \tilde{\mu}^\top(\tilde{\Sigma}^\perp)^{-1}\tilde{\mu}\right) \\
&= \frac{1}{2A}\min_{\rho_{ij}, 1\le i<j<N}\left(\min_{\rho_{iN}, 1\le i<N}(\frac{\tilde{\alpha}_N^2}{\tilde{\sigma}_N^2}) + \tilde{\mu}^\top(\tilde{\Sigma}^\perp)^{-1}\tilde{\mu}\right),
\end{aligned}$$

with the optimal weight

$$\psi^* = \frac{1}{A}\begin{pmatrix} (\tilde{\Sigma}^\perp)^{-1}\tilde{\mu} \\ \tilde{\sigma}_N^{-2}\tilde{\alpha}_N \end{pmatrix}|_{\rho=\rho^*}.$$

So if asset N is not held, $\tilde{\sigma}_N^{-1}\tilde{\alpha}_N|_{\rho=\rho^*} = 0$ we must have $\min_\rho \tilde{\alpha}_N^2 = 0$ [7]. Conversely, if

$$\min_{\rho_{iN}, 1 \le i < N} \tilde{\alpha}_N^2 / \tilde{\sigma}_N^2 = 0$$

or

$$\min_{\rho_{iN}, 1 \le i < N} \tilde{\alpha}_N^2 = 0$$

holds for all $\tilde{\Sigma}^{\perp}$ (i.e. for all $\rho_{ij}, 1 \le i < j < N$), there is no reason to hold asset N which has an zero Sharpe ratio in the worst case. □

Proof of Corollary 1 and Proposition 4:

Exploiting the variable transformation $\psi = \sigma\phi$ and the minimax theorem, we obtain

$$J = \max_{\psi} \min_{\rho_{ij} \in [\underline{\rho}_{ij}, \bar{\rho}_{ij}]} s^\top \psi - \frac{A}{2}\psi^\top \rho \psi = \min_{\rho_{ij} \in [\underline{\rho}_{ij}, \bar{\rho}_{ij}]} \max_{\psi} s^\top \psi - \frac{A}{2}\psi^\top \rho \psi.$$

It is trivial to solve the inner maximization problem in the right hand side above. We have $\psi^* = \frac{1}{A}\rho^{-1}s$, and $J = \min_\rho \frac{1}{2A}s^\top \rho^{-1}s$. Let $f = \frac{1}{2A}s^\top \rho^{-1}s$. The first order condition of f w.r.t. ρ_{ij} is

$$\frac{\partial f}{\partial \rho_{ij}} = \frac{-1}{2A}s^\top \rho^{-1} I_{ij} \rho^{-1} s = -\frac{A}{2}((\psi^*)^\top I_{ij}(\psi^*)) = -A\psi_i^*\psi_j^*,$$

where I_{ij} is a $N \times N$ matrix with all zero entries except 1 at the entry (i, j) and (j, i). If the minimization is achieved at an interior point of $[\underline{\rho}_{ij}, \bar{\rho}_{ij}]$, then $\partial f / \partial \rho_{ij} = 0$. As a result, $\psi_i^* \psi_j^* = 0$. This completes the proof. □

[7]Since Σ must be positive definite, $\tilde{\sigma}_N^2$ must be positive. We may assume that $\tilde{\sigma}_N^2$ is away from 0 for given ambiguous sets of correlations.

Appendix B. Transforming to SDP

Note that

$$\min_{\rho} \max_{\phi} \phi^\top s - \frac{A}{2}\phi^\top \rho \phi = \min_{\rho} \frac{1}{2A} s^\top \rho^{-1} s.$$

We need to solve the minimization problem $\min_\rho s^\top \rho^{-1} s$. This minimization problem is not a standard semi-definite programming problem yet. We take a transformation as follows. Consider

$$\begin{aligned} (P1): &\quad \min s^\top \rho^{-1} s, \\ s.t. &\quad \rho \in [\underline{\rho}, \overline{\rho}], \qquad \rho > 0, \end{aligned}$$

where $\rho > 0$ denotes the positive definite constraint. The problem (P1) can be rewritten as follows.

$$\begin{aligned} (P1'): &\quad \min t, \\ s.t. &\quad s^\top \rho^{-1} s \le t, \quad \rho \in [\underline{\rho}, \overline{\rho}] \quad \rho > 0. \end{aligned}$$

We claim that the constraint $s^\top \rho^{-1} s \le t$ and $\rho \ge 0$ is equivalent to $\begin{bmatrix} \rho & s \\ s^\top & t \end{bmatrix} \ge 0$ because

$$\begin{bmatrix} \rho & s \\ s^\top & t \end{bmatrix} \ge 0 \Longleftrightarrow \begin{bmatrix} \rho & 0 \\ 0 & t - s^\top \rho^{-1} s \end{bmatrix} \ge 0.$$

So, (P1) can be transformed to:

$$\begin{aligned} (P2): &\quad \min t, \\ s.t. &\quad \begin{bmatrix} \rho & s \\ s^\top & t \end{bmatrix} \ge 0, \quad \rho \in [\underline{\rho}, \overline{\rho}], \quad \rho > 0. \end{aligned}$$

(P2) is a standard SDP problem.

Semidefinite programming (SDP) is a subfield of convex optimization for a linear objective function over the intersection of the cone of positive semidefinite matrices with an affine space. SDP is a special case of cone programming and can be solved by interior point methods. There are many free codes available in various programming languages, for example, C, C++, Matlab, Python. In the paper, we use Yalmip toolbox[8] with DSDP solver, developed by Steve Benson, Yinyu Ye, and Xiong Zhang. For a complete description of the algorithm and a proof of convergence of DSDP, see "Solving Large-Scale Sparse Semidefinite Programs for Combinatorial Optimization", SIAM Journal on Optimization, 10(2), 2000, pp. 443-461.

Appendix C. A Note on Minimax Theorem

We apply the minimax result in Theorem 37.3 of Rockafellar (1970) to establish the relation of $\max_\phi \min_\rho$ and $\min_\rho \max_\phi$ in equation (3) and occasionally elsewhere of this paper. In fact, Corollary 37.6 of Rockafellar (1970) further indicates that there exists a saddle point solution (π^*, ϕ^*) such that

$$\Phi(\phi, \rho^*) \leq \Phi(\phi^*, \rho^*) \leq \Phi(\phi^*, \rho)$$

for any ϕ and ρ, where $\Phi(\phi, \rho) = \mu^\top \phi - \frac{A}{2} \phi^\top \Sigma \phi$ is a concave-convex function. Hence the optimal solution discussed in this paper is actually a saddle point solution.

However, we need both of ϕ and ρ are chosen from bounded sets in order to apply the theorem and corollary. Correlations are chosen from uncertain sets (with a nonempty interior) which are naturally bounded by -1 and 1, while the portfolio weight vector ϕ may be required in a large enough but bounded set of $\mathcal{R}^N$. Precisely, we may require $\phi \in [-M, M]^N \in \mathcal{R}^N$, where M is greater than the maximal absolute value of components of the vector $\frac{1}{A}\Sigma^{-1}(\rho)\mu$ over all ρ. Then the boundedness assumption is satisfied.

We refer to Halldorsson and Tütüncü (2003) for a related article and Kim and Boyd (2008) for an alternative approach to the minimax result without the boundedness assump-

[8]The toolbox is available at http://users.isy.liu.se/johanl/yalmip/pmwiki.php?n=Main.WhatIsYALMIP

tion.

In this paper, we use the minimax result for easy exposition and clear economic intuition. It worths mentioning that our major results in this paper do not reply on the minimax theorem. For example, for the case of two risky assets, we can use the argument that directly solves the max-min problem as follows.

Consider three cases: Case 1: $\phi_1^* \phi_2^* > 0$. The optimal correlation for the inner minimization problem is $\rho^* = \bar{\rho}$. Then we can solve the outside maximization problem and find $\phi_1^* = \frac{1}{A}(s_1 - \bar{\rho} s_2)$ and $\phi_2^* = \frac{1}{A}(s_2 - \bar{\rho} s_1)$. To comply with the condition $\phi_1^* \phi_2^* > 0$, we must have $\bar{\rho} < s_2/s_1$. Together with further analysis on Case 2 of $\phi_1^* \phi_2^* < 0$ and Case 3 of $\phi_1^* \phi_2^* = 0$, we arrive at Proposition 2, without changing the order of max and min. Also we can show anti-diversification in the N-asset case directly and compute empirical optimal portfolios by a straight numerical approach.

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Figures and Tables

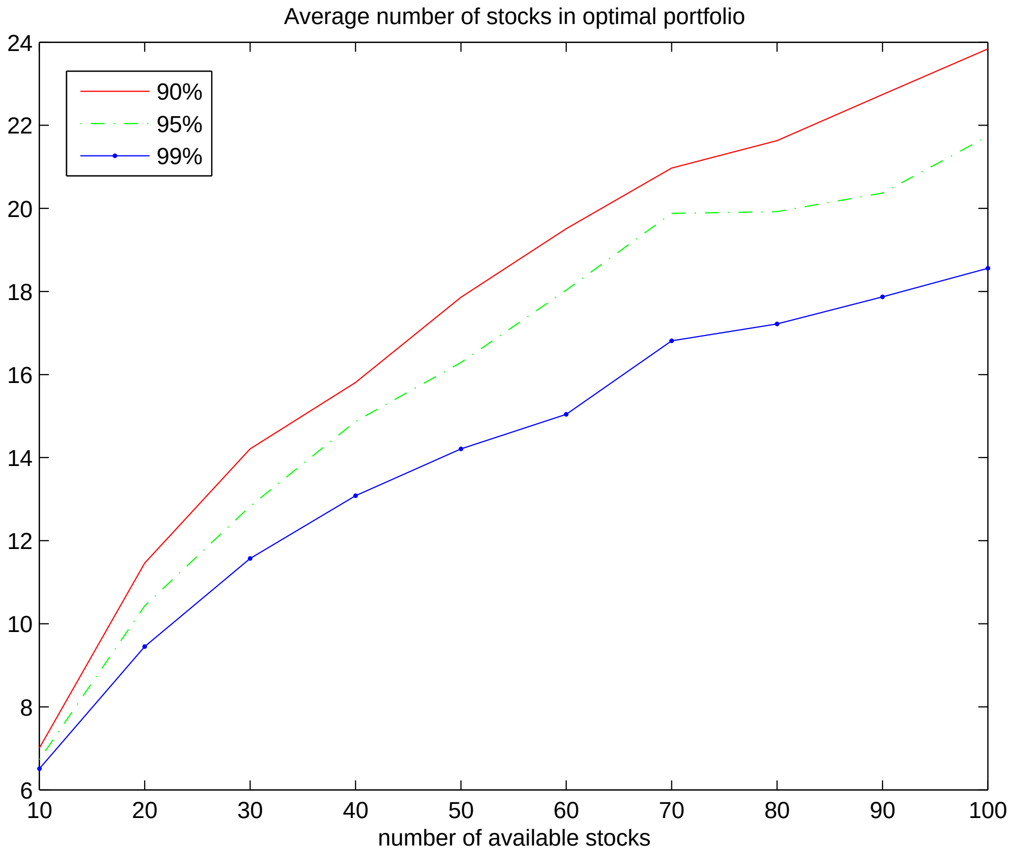

Figure 1. We randomly choose N $(= 10, 20, ..., 100)$ stocks from the S&P 500, and find the optimal portfolio under correlation ambiguity for these N stocks. The ambiguous sets are given by the 90%, 95%, and 99% confidence intervals respectively. We repeat the procedure 100 times, average number of stocks in optimal portfolios for each N and for each ambiguous level is calculated and reported in the figure. The X-axis denotes number of available stocks (N) varying from 10 to 100. The Y-axis denotes average number of stocks in optimal portfolios.

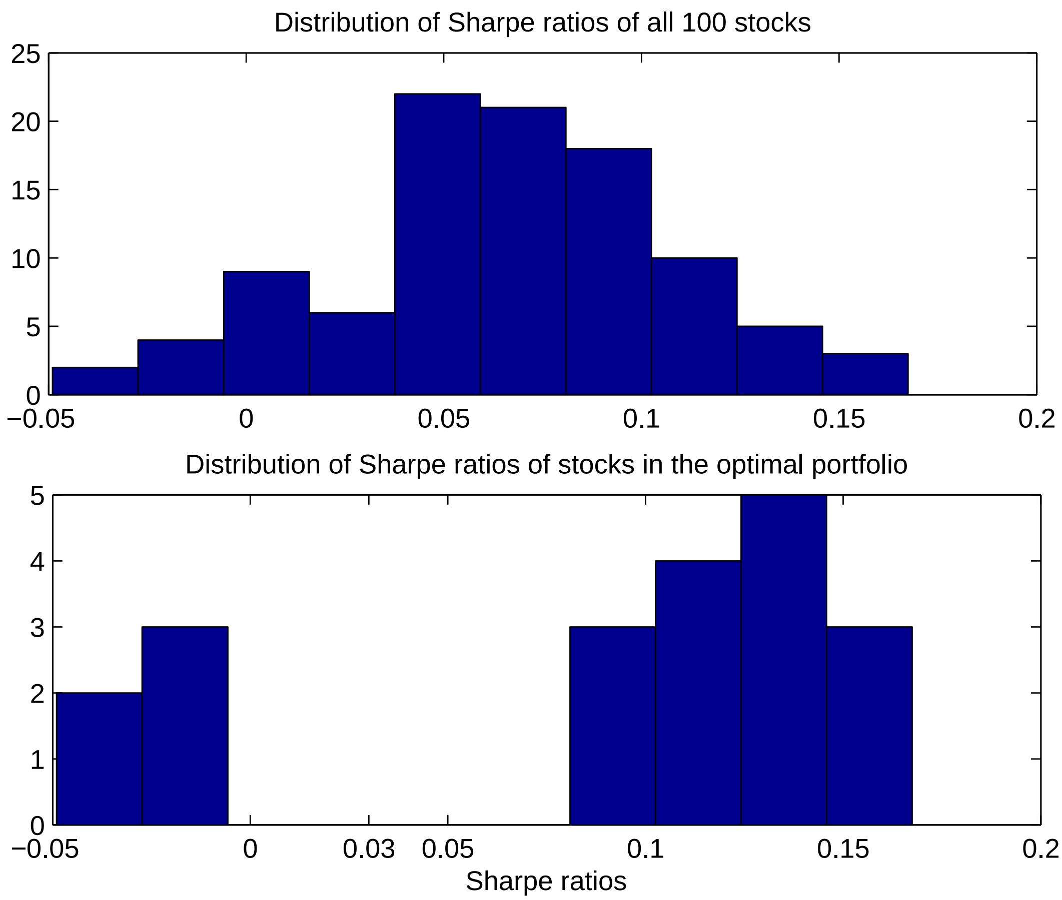

Figure 2. This figure shows distributions of Sharpe ratios in one randomly-selected sample of 100 stocks, as well as in the optimal portfolio from these stocks. Although many stocks with low Sharpe ratios are not held in the optimal portfolio, a few stocks with low Sharpe ratios are still held. Correlations and ambiguous levels matter as well as Sharpe ratios. By Proposition 2, roughly speaking, a stock will not be held in the optimal portfolio if the ratio of its Sharpe ratio to the maximal Sharpe ratio falls into the ambiguous interval of the correlation. Hence given the average correlation of 0.2220 and the maximal Sharpe ratio of 0.1674, stocks with a Sharpe ratio of approximate $0.0372 (= 0.222 \times 0.1674)$ should tend not to be held. This analysis is consistent with the figure, which shows that stocks with a Sharpe ratio near 0.03 are not held.

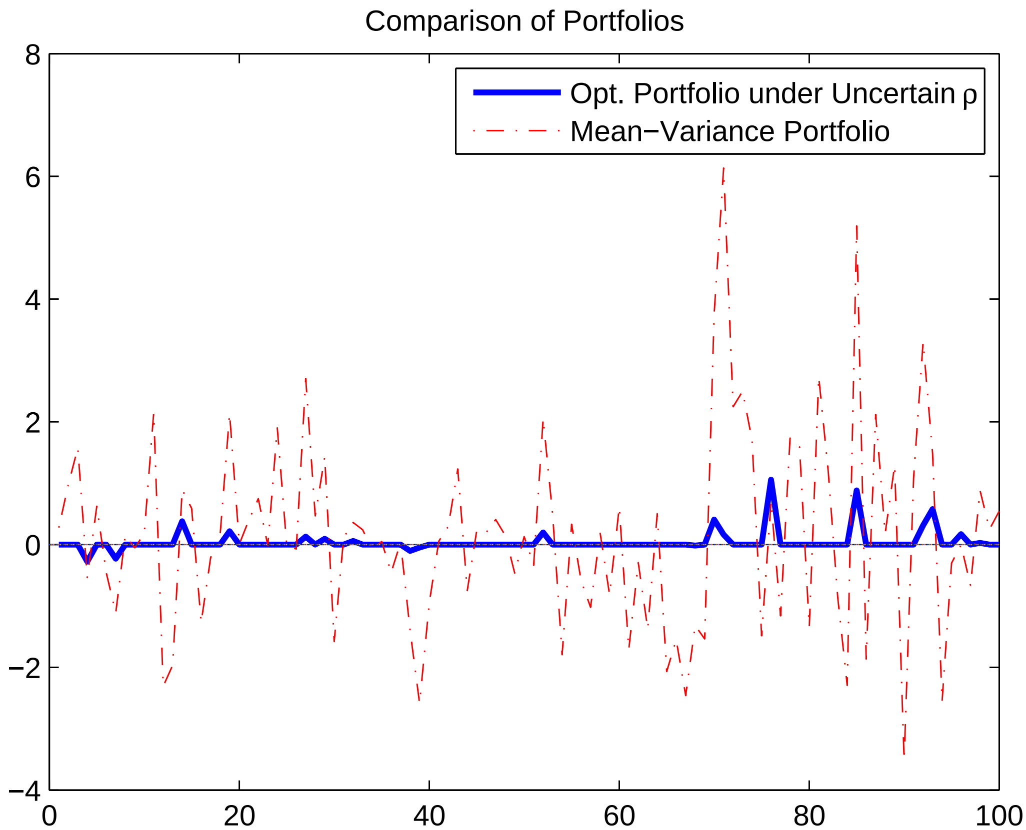

Figure 3. The magnitude of large positions are reduced greatly in the optimal portfolio compared to the standard mean-variance portfolio based on the estimation $\hat{\rho}$. A total of 100 stocks are in the sample. Every integer on the X-axis represents a stock. The Y-axis denotes allocations on stocks. The risk aversion coefficient A is 1. The extreme positions in the standard mean-variance portfolio are significantly reduced in the optimal portfolio under correlation ambiguity. Consequently, the optimal portfolio has less variance and is more conservative than the mean-variance portfolio in this sense.

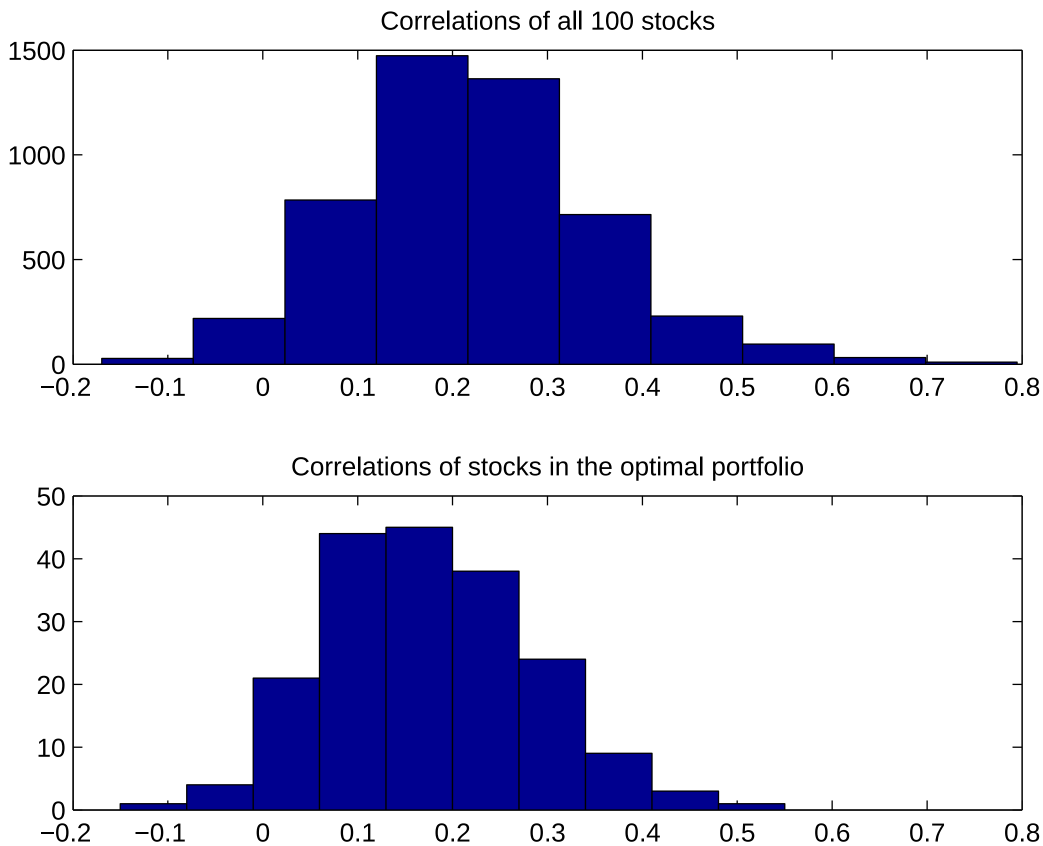

Figure 4. This figure shows the distributions of the correlations of all 100 stocks, as well as the stocks in the optimal portfolio. There are a total of $4950 (= 100 \times 99/2)$ correlations in the top panel, whereas there are 190 $(= 20 \times 19/2)$ correlations in the bottom panel. The top correlation as well as many high correlations disappear in the optimal portfolio. Consequently, less extreme positions are found in the optimal portfolio.

Table I. Statistics of Correlations, Mean, and Sharpe Ratios of Excess Returns

level	correlations				length of C.I.			
of C.I.	mean	std	max	min	mean	std	max	min
99%	0.2369	0.1266	0.8365	-0.6385	0.3088	0.0227	0.3322	0.1016
95%					0.2357	0.0175	0.2537	0.0770
90%					0.1980	0.0148	0.2133	0.0645
	excess returns				Sharpe ratios			
	mean	std	max	min	mean	std	max	min
	0.0051	0.0042	0.0173	-0.009	0.060	0.0443	0.1829	-0.0762

The total number of stocks is 319. The percentage of p-values greater than 0.05 is 0.18%. Hence most correlations are significant. Average monthly LIBOR is used as the risk free return.

Table II. Comparison of Sharpe Ratios between Held and Not-held Stocks

level of C.I.	95%		99%	
	Held	Not-held	Held	Not-held
Number of Stocks	20	80	16	84
max. Sharpe	**0.1674**	0.1153	0.1674	0.1125
min. Sharpe	-0.0489	-0.0064	-0.0489	-0.0129
ave. Sharpe	0.0859	0.0600	0.0954	0.0594

The table lists the results for 100 randomly selected stocks (with average correlation of 0.2220). The numbers of stocks held in the optimal portfolios are 20 and 16, and the numbers of stocks not-held are 80 and 84, respectively, for two levels of ambiguity. At each level of ambiguity, average Sharpe ratios are reported in the table, as well as the maximum and minimum of Sharpe ratios. Basically, held stocks have higher Sharpe ratios on average than not-held stocks.

Table III. Comparison of Non-zero Positions

OP	-0.2776	-0.2288	0.3792	0.2185	0.1289	0.0964	0.0584	-0.1040	**-0.0483**	0.0013
	0.1986	-0.0175	0.4063	**0.1604**	1.0580	0.8835	0.3129	0.5768	0.1715	0.0252
MV	-0.5354	-1.1119	0.8954	2.1356	2.7047	1.4276	0.3582	-1.3984	**-2.5868**	0.4069
	2.0538	-1.3190	3.7618	**6.1355**	0.7032	5.1949	3.3031	1.4833	-0.0082	0.8841

"OP" represents the optimal portfolio under correlation ambiguity. "MV" represents the corresponding positions in the standard mean-variance portfolio. As shown in the table, the extreme positions (in bold fonts) in the MV are significantly reduced in the optimal portfolio under correlation ambiguity. The risk aversion A is 1.