Keynes Meets Markowitz: The Tradeoff Between Familiarity and Diversification*

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This version: February 14, 2009

*We thank Suleyman Basak, Bob Goldstein, Francisco Gomes, Dietmar Leisen, and Anthony Neuberger for their extensive suggestions. We also acknowledge comments from Ron Anderson, Juan-Pedro Gomez, Gur Huberman, Vasant Naik, Hongjun Yan and from participants at seminars at Lehman Brothers (London), McGill University, Northwestern University, Oxford University, University of Southern Switzerland, University of Toronto, Wilfrid Laurier University, Yale University, the CEPR Conference on Financing Retirement in Europe, the Northern Finance Association, the International Symposium on Insurance and Finance at Bergen and the 2005 Meetings of the Western Finance Association.

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February 14, 2009

Abstract

We develop a model of portfolio choice capable of nesting the views of Keynes, advocating concentration in a few familiar assets, and Markowitz, advocating diversification across all available assets. In the model, the return distributions of risky assets are ambiguous, and investors are averse to this ambiguity. The model allows for different degrees of "familiarity" for various assets and captures the tradeoff between concentration and diversification. The models shows that if investors are not familiar about a particular asset, then they hold a diversified portfolio, as advocated by Markowitz. On the other hand, if investors are familiar about a particular asset, they tilt their portfolio toward that asset, while continuing to diversify by holding the other assets in the market. And, if investors are familiar about a particular asset *and* sufficiently ambiguous about all other assets, then they hold only the familiar asset, as Keynes would have advocated. Finally, if investors are sufficiently ambiguous about all risky assets, then they will not participate at all in the equity market. The model shows that even when the number of assets available for investment is very large, investors continue to hold familiar assets, and an increase in correlation between familiar assets and the rest of the market leads to an increase in the allocation to familiar assets, as we observe empirically during periods of financial crises. We also show how β and the risk premium of stocks can depend on both systematic and unsystematic volatility. Our model predicts also that, when the aggregate level of ambiguity in the economy is large, investors increase concentration in the assets with which they are more familiar (flight to *familiarity*), even if these happen to be assets with a higher risk or lower expected return.

Keywords: Investment, portfolio choice, ambiguity, robust control,
JEL Classification: G11, G12, G23, D81

1 Introduction

John Maynard Keynes and Harry Markowitz have very different views about portfolio selection. Markowitz (1952, p. 77) argues that it is inefficient to put a large holding in just a few stocks, and that investors should diversify across a large number of stocks:

> "Diversification is both observed and sensible; a rule of behavior which does not imply the superiority of diversification must be rejected both as a hypothesis and as a maxim."

Even though Markowitz's idea of diversification has been accepted as one of the most fundamental tenets of modern financial economics, empirical evidence suggests that investors do not hold diversified portfolios but rather invest heavily in only a few assets,[1] and typically those with which they are *familiar*.[2]

However, Keynes in a letter written in 1934 expresses the view that one should allocate wealth in the few stocks about which one feels most favorably:

> "As time goes on I get more and more convinced that the right method in investment is to put fairly large sums into enterprises which one thinks one knows something about and in the management of which one thoroughly believes. It is a mistake to think that one limits one's risk by spreading too much between enterprises about which one knows little and has no reason for special confidence. [...] One's knowledge and experience are

[1]Early evidence on the lack of diversification across a large number of assets is provided by Blume and Friend (1975); using data on income tax returns, they find that most investors hold only one or two stocks. Kelly (1995), relying on data from the Survey of Consumer Finances (SCF), finds that mean-variance efficiency does not describe very well the portfolio diversification of U.S. households: the median stockholder owns a single publicly traded stock, often in the company where he works. Polkovnichenko (2005)), also using data from SCF, finds that of the households that hold individual stocks directly, the median number of stocks held was two from 1983 until 2001, when it increased to three; and, poor diversification is often attributable to investments in employer stock, which is a significant part of equity portfolios. Barber and Odean (2000) and Goetzman and Kumar (2008) report similar findings of under-diversification based on data for individual investors at a U.S. brokerage. Calvet, Campbell, and Sodini (2007), based on detailed government records covering the entire Swedish population, find that some households are poorly diversified and bear significant idiosyncratic risk. A comprehensive summary of this aspect of household finance is provided in Campbell (2006, Section 3).

[2]Huberman (2001) introduced the idea that people invest in *familiar* assets and provides evidence of this in a multitude of contexts; for example, he documents that in most U.S. states the investment in local regional Bell phone companies far exceeds the amount invested in out-of-state Bell companies. Massa and Simonov (2006) also find that investors tilt their portfolios away from the market portfolio and toward stocks that are geographically and professionally close to the investor, resulting in a portfolio biased toward familiar stocks. French and Poterba (1990), Cooper and Kaplanis (1994), and Tesar and Werner (1995) document that investors bias their portfolios toward "home equity" rather than diversifying internationally; Grinblatt and Keloharju (2001) and Massa and Simonov (2006) find this bias to be present among Finnish and Swedish investors, respectively; and, Feng and Seasholes (2004) find that not only do Chinese investors overweight local companies but also companies that are traded on a local exchange. Coval and Moskovitz (1999) show that the bias toward familiar assets is not just in international portfolios of small investors, but also U.S. investment managers have a bias toward local equities that are geographically close to the managers. Sarkissian and Schill (2004)) find that the familiarity bias is not just in investment decisions but also in financing decisions: the decision to cross list equity in a particular foreign market is driven by how familiar is that market rather than diversification considerations. A good summary of this literature is provided in Vissing-Jorgensen (2003, Section 4.2).

> definitely limited and there are seldom more than two or three enterprises at any given time in which I personally feel myself entitled to put full confidence."[3]

And, this is not just an outdated view: Warren Buffet espouses the same view, and included in his letter to shareholders of Berkshire Hathaway in 1991 this very quote from Keynes.[4]

The academic literature, however, has paid little attention to Keynes's view on portfolio selection. The almost exclusive attention to Markowitz's theory is perhaps due to its elegant analytic representation, a feature which is absent for Keynes' view. Our objective in this paper is to develop a model that allows us to assess *quantitatively* the different trade-offs advocated by Keynes and Markowitz. In particular, we wish to understand the implications of Keynes's view for the portfolio selected by individual investors, and answer the following kind of questions. Under what circumstances should investors hold only the assets with which they are familiar, and when should they hold also the market portfolio? When they hold both, how much should they invest in familiar assets and how much in the market? How does the volatility of individual assets and the correlation between familiar assets and the market affect the relative allocation between these two? If the number of assets available for investment is very large, does the gain from diversification overwhelm the motivation to invest in familiar assets? When investors do not diversify perfectly, as Markowitz would have recommended, what affect does this have on the beta of stocks and their risk premium in equilibrium?

We model lack of familiarity via the concept of *ambiguity* (or uncertainty) in the sense of Knight (1921). Investors are ambiguous about the distribution of returns of assets, and can have different degrees of ambiguity for different assets. Specifically, we build on the portfolio selection framework of an ambiguity averse investor developed in Garlappi, Uppal, and Wang (2006).[5] The framework we develop has three attractive features: (1) it is simple and only a mild departure from the well-understood Markowitz (1959) model, (2) it has a strong axiomatic foundation, and (3) yet it is capable of capturing some of the observed evidence on household portfolios. This framework can also be thought of as providing a mathematical representation of the concept of familiarity introduced by Huberman (2001). Such a framework is especially relevant for investment decisions given the finding of Heath and Tversky (1991) that ambiguity aversion is particularly

[3]From Keynes' letter to F. C. Scott, 15 August, 1934, Keynes (1983). Keynes was not alone in holding such a view. Loeb (1950), for example, advocated that, "Once you obtain confidence, diversification is undesirable; diversification [is] an admission of not knowing what to do and an effort to strike an average."

[4]This letter is available at http://www.berkshirehathaway.com/letters/1991.html.

[5]Our model is related to other models founded on Gilboa and Schmeidler (1989), which include: Dow and Werlang (1992), Epstein and Wang (1994), Hansen, Sargent, and Tallarini (1999), Anderson, Hansen, and Sargent (2003), Hansen and Sargent (2000), Chen and Epstein (2002), Epstein and Miao (2003), Uppal and Wang (2003), and Maenhout (2004). The relation between these various models is discussed in Hansen and Sargent (2001a), Hansen and Sargent (2001b) and Schroder and Skiadas (2003).

strong in cases where people feel that their competence in assessing the relevant probabilities is low. Fox and Tversky (1995) also show that a significant percentage of people demonstrate ambiguity aversion in a *comparative situation*, which is one where prospects of different degrees of ambiguity are contrasted, as opposed to being considered in isolation. Our view is that this is exactly the situation in a portfolio-choice setting, where assets of different degrees of ambiguity are considered together in the allocation decision.[6]

Our model allows us to derive the following *analytic* results that address the questions listed above. If investors are not familiar about a particular asset, then they hold a diversified portfolio, as advocated by Markowitz. On the other hand, if investors are familiar about a particular asset, they tilt their portfolio toward that asset, while continuing to invest in other assets; that is, there is both concentration in the more familiar asset *and* diversification in other assets; consequently, this portfolio will bear idiosyncratic risk in addition to systematic risk. If investors are familiar about a particular asset *and* sufficiently ambiguous about all other assets, then they will hold *only* the familiar asset, as Keynes would have advocated. Moreover, if investors are sufficiently ambiguous about all risky assets, then they will not participate at all in the equity market. The model also shows that when the level of average ambiguity across all assets is low, then the relative weight in the familiar asset decreases as its volatility increases; but the reverse is true when the level of average ambiguity is high. An increase in correlation between familiar assets and the rest of the market leads to a reduction in the investment in the market. And, even when the number of assets available for investment is very large, investors continue to hold familiar assets. When investors hold only familiar assets, we show how the beta of stocks depends on both systematic and unsystematic volatility, and we derive the relation between expected stock returns and systematic and unsystematic risk in equilibrium.

We conclude this introduction by mentioning that information advantage is commonly offered as a potential explanation for under-diversification in portfolio choice. That is, investors who believe that the expected return of an asset is high, will invest more in that asset. While appealing, an information-based explanation cannot explain entirely the observed bias toward familiar stocks, such as the "own-company puzzle" and "home bias puzzle".[7] First, this explanation will hold true only if the investor believes that the expected return on the familiar asset (for example, own-

[6]Polkovnichenko (2005) provides a different model that is based on rank-dependent preferences rather than familiarity. Liu (2008) argues that if investors require a nonnegative level of consumption at which marginal utility is finite, and do not buy on margin or short sell, then they will underdiversify, especially when wealth is low; he also finds that idiosyncratic risk can be priced in equilibrium.

[7]Recently, van Nieuwerburgh and Veldkamp (2008a) develop a model of information *acquisition* and portfolio choice which is capable of explaining portfolio concentration by exploiting the increasing return to scale to specializing in one asset. They use this model in van Nieuwerburgh and Veldkamp (2005) to study the bias toward investing in own-company stock and extend this to a two-country general equilibrium setting in van Nieuwerburgh and Veldkamp (2008b) to study the "home-bias" puzzle.

company stock or domestic equity) to always be higher than that on other assets; but, information about the familiar asset can also be negative, in which case the information-based explanation would recommend a *negative* position in the familiar asset. Second, an information-based explanation, per se, will not be capable of explaining why investors often do not invest at all in most of the available assets. Finally, an information-based explanation would indicate that investors who hold portfolios that are biased toward a few assets outperform diversified portfolios, but the empirical evidence on this is mixed—for example, Calvet, Campbell, and Sodini (2007) and Goetzman and Kumar (2008) do not find support for this, while Ivković and Weisbenner (2005) and Massa and Simonov (2006) find evidence that investors may have superior information about familiar assets.

The remainder of this paper is organized as follows. In Section 2, we develop the theoretical framework that allows us to study the tradeoff among return, risk, and familiarity. In Section 3, we derive analytically the implications of this framework for optimal portfolio choice in the presence of ambiguity about the distributions for asset returns. Our conclusions are presented in Section 4. Our main results are highlighted in propositions, and the proofs of propositions are relegated to the Appendix.

2 A Model that Nests the Views of Keynes and Markowitz

In this section, we introduce a model of portfolio choice that incorporates an investor's ambiguity about the true distribution of asset returns. We first describe our assumptions about the statistical properties of asset returns. We then formulate the portfolio problem of an investor in the classical setting of Markowitz (1959). We conclude this section by formulating the portfolio problem of an investor who is averse to ambiguity.

We have made a conscious decision to consider a static, discrete-time model model where all assets have the same expected return, volatility, and correlation with other assets, with the only difference among assets being the degree of ambiguity an investor has about their returns. Our choice of a static model set in discrete-time dictated by the desire for simplicity; our choice of identical asset returns is driven by the desire to focus on the key characteristic of our framework: differences in familiarity (or ambiguity) across assets.[8]

[8]The issue of portfolio choice in a dynamic setting when the agent is averse to ambiguity has already been addressed in the literature. The case of dynamic portfolio choice with only a single risky asset in a robust-control setting is addressed in Maenhout (2004); the case of multiple risky assets in a dynamic setting in which investors are averse to ambiguity is considered in Chen and Epstein (2002), Epstein and Miao (2003), and Uppal and Wang (2003).

2.1 Asset returns

We consider a static economy with N identical risky assets. The return on each risky asset, in excess of the risk-free rate, is

$$r_n = r_S + u_n, \quad n = 1, \ldots, N, \tag{1}$$

in which r_S represents the *systematic* component of individual returns and u_n is a zero-mean random variable capturing the *unsystematic* (or idiosyncratic) component of individual returns. We denote by μ_S and σ_S the expected return and volatility, respectively, of the systematic component r_S. For simplicity, we assume that the unsystematic components of any two assets are uncorrelated, that is, $\mathrm{Cov}(u_n, u_k) = 0$ for all n and k. We also assume that the volatility σ_U of u_n to be identical across assets.

Given the above assumptions, the expected return on Asset n is: $\mu_n = \mu_S$. Letting $\mathbf{1}_N$ denote a $N \times 1$ vector of ones, and $\mathbf{1}_{N\times N}$ denote the N-dimensional identity matrix, the N-dimensional vector of expected excess returns is given by

$$\boldsymbol{\mu} = \mu_S \, \mathbf{1}_N, \tag{2}$$

and the $N \times N$ variance-covariance matrix is

$$\Sigma = \sigma_U^2 \, \mathbf{1}_{N\times N} + \sigma_S^2 \, \mathbf{1}_N \mathbf{1}_N^\top = \begin{bmatrix} \sigma_S^2 + \sigma_U^2 & \sigma_S^2 & \cdots & \sigma_S^2 \\ \sigma_S^2 & \sigma_S^2 + \sigma_U^2 & \cdots & \sigma_S^2 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_S^2 & \sigma_S^2 & \cdots & \sigma_S^2 + \sigma_U^2 \end{bmatrix}.$$

Observe that the common variance of each asset is given by

$$\sigma_n^2 \equiv \sigma^2 = \sigma_S^2 + \sigma_U^2, \tag{3}$$

and the common correlation across any two assets is given by

$$\rho_{nk} \equiv \rho = \frac{\sigma_S^2}{(\sigma_S^2 + \sigma_U^2)} = \frac{\sigma_S^2}{\sigma^2} > 0. \tag{4}$$

Note also that this simplified specification implies that the return on all risky assets must exceed that on the risk-free asset, and therefore, the Sharpe ratio must be positive: $\mu_n/\sigma_n > 0$.

By focusing on assets with identical return distributions, we are able to isolate the effect of ambiguity on portfolio choice; however, it is straightforward to extend the model to a setting where each asset has distinct first and second moments.

2.2 The Markowitz mean-variance portfolio problem and optimal weights

The investor's problem is to choose a vector of portfolio weights, $\pi = (\pi_1, \pi_2, \ldots, \pi_N)$, for the available N risky assets. According to the classical mean-variance model (see Markowitz (1952, 1959), and Sharpe (1970)), the optimal portfolio of a risk averse investor is given by the solution to the following optimization problem,

$$\max_{\pi} \; \pi^\top \boldsymbol{\mu} - \frac{\gamma}{2} \pi^\top \Sigma \, \pi, \tag{5}$$

where γ denotes the investor's risk-aversion parameter. If the investor knows the true parameters μ and Σ, the solution to this problem is

$$\pi = \frac{1}{\gamma} \Sigma^{-1} \boldsymbol{\mu} = \frac{1}{\gamma} \left(\frac{\mu_S}{\sigma^2 \left(1 + \rho(N-1)\right)} \right) \mathbf{1}_N, \tag{6}$$

where the last equality follows from imposing the properties of returns described in Section 2.1. For expositional ease, it is convenient to compute also the *relative* portfolio weights, that is, the fraction of the risky portfolio invested in each asset. Formally, this quantity is defined as $\omega = \pi/(\pi^\top \mathbf{1}_N)$. Because all the assets are assumed to be identical in terms of risk and return, the Markowitz portfolio in Equation (6) obviously suggests holding an identical amount in each of the assets, and thus, the relative portfolio weight in each asset is equal to $1/N$.

2.3 The "ambiguity-averse" portfolio problem

A fundamental assumption of the standard mean-variance portfolio selection problem formulated in (5) is that the investor knows the true expected returns. In practice, however, the investor has to estimate expected returns, a notoriously difficult task (see Merton (1980)). The difficulty in estimating precisely the expected returns introduces the possibility of ambiguity about the true distribution of the asset returns into the investor's portfolio choice problem.

We assume that agents do not have precise knowledge of the distribution of the returns of the assets in the economy and that they are *averse* to such ambiguity.[9] Moreover, in general, investors may have different ambiguity about different assets.

An intuitive way of capturing such ambiguity is to rely on results from classical statistics and consider the confidence interval for the estimator of the expected returns. We focus on the error in

[9] In the literature, parameter uncertainty and estimation risk are also used to describe the problem the investor faces. Parameter uncertainty is often dealt with using a Bayesian approach in which the unknown parameters are treated as random variables that are "integrated out" while maximizing utility in order to find optimal portfolios. A Bayesian investor is *neutral* to ambiguity, since he is capable of aggregating the uncertainty about the parameters via a subjective prior. We are interested in the case of investors who *cannot* form a unique prior on the uncertain parameters and are averse to this ambiguity. We adopt this model in contrast to the Bayesian approach because our model allows us to generate zero holdings in some assets, a result that cannot be obtained in a Bayesian model.

estimating expected returns of assets because, as shown in Merton (1980), they are much harder to estimate than the variances and covariances. Let $\hat{\mu}_n$ be the estimated value of the mean return of asset n obtained by using a return time series of length T, and

$$\hat{\sigma}_n \equiv \sqrt{\sigma_n^2/T}, \tag{7}$$

the standard deviation of the estimate $\hat{\mu}_n$.[10] If excess asset returns are normally distributed and σ_n is known, the quantity $(\hat{\mu}_n - \mu_n)/\hat{\sigma}_n$ has a standard Normal distribution. We can hence define the following *confidence interval* for the expected return μ_n:

$$\alpha_n\text{-confidence interval} = \left\{ \mu_n \; : \; \frac{(\mu_n - \hat{\mu}_n)^2}{\hat{\sigma}_n^2} \leq \alpha_n^2 \right\}, \tag{8}$$

where α_n is the critical value determining the size, or level, of the confidence interval. The above expression suggests that the critical level α_n can be directly interpreted as a measure of the amount of ambiguity the investor faces when presented with the outcome of a statistical estimate of the expected returns. A larger value of α_n will result in a larger confidence interval, and hence, a larger set of possible distributions to which the true returns may belong.[11]

If historical data is all the information an investor has, then the above confidence interval, which is based on historical data, is a good description of the ambiguity about the returns on that asset. If, in addition, the investor has further knowledge about a particular asset, say, through working for that company, then this familiarity would typically lead to a reduction in the ambiguity about this asset. Mathematically, this can be represented by a smaller α_n in (8). A smaller α_n is the result of familiarity *above and beyond* the information contained in historical data on returns.

To capture both the presence of ambiguity and the aversion to it, we rely on the work of Gilboa and Schmeidler (1989) using the approach adopted in Garlappi, Uppal, and Wang (2006) by introducing two new components into the standard mean-variance portfolio selection problem in (5). First, we model the *presence* of ambiguity by imposing the confidence interval (8) as an *additional constraint* on the mean-variance optimization program. Second, to account for *aversion* to ambiguity, we impose that the the investor chooses his portfolio by *minimizing* over the set of expected returns he considers plausible according to (8).[12] Thus, the extended mean-variance

[10] We consider the case where the same length over which the mean return is estimated is the same for all assets. It is straightforward to extend the model to allow for different lengths of time over which the mean returns for different assets are estimated.

[11] Bewley (1988) originally formulated the argument that confidence intervals can be interpreted also as a measure of the level of ambiguity associated with the estimated parameters. For another recent paper that uses Bewley's characterization of Knightian uncertainty, see Easley and O'Hara (2008).

[12] Klibanoff, Marinacci, and Mukerji (2005) develop an alternative model of decision making under uncertainty; in contrast to the maxmin specification that we use, which generates kinked demand functions, their model generates smooth demand functions.

model takes the following form:

$$\max_{\pi} \min_{\boldsymbol{\mu}} \; \pi^{\top}\boldsymbol{\mu} - \frac{\gamma}{2}\pi^{\top}\Sigma\pi, \tag{9}$$

subject to

$$\frac{(\mu_n - \hat{\mu}_n)^2}{\hat{\sigma}_n^2} \leq \alpha_n^2, \quad n = 1, \ldots, N, \tag{10}$$

where α_n refers to the level of ambiguity associated with asset n. In the extreme case in which $\alpha_n = 0$ for all n, the optimal portfolio in (9) reduces to the mean-variance portfolio problem in (5) in which the estimated means, $\hat{\mu}_n$, are used as values for the expected returns, μ_n.

3 The optimal portfolio weights and their interpretation

In this section, we solve for the optimal portfolio weights of an investor who is familiar toward one or more assets, and then explain how the optimal weights can be interpreted in terms of the views of both Markowitz and Keynes, and the implications of these weights for aggregate quantities such as the market beta and the market risk premium.

To understand the solution to the problem in Equations (9) and (10), and to highlight the Markowitz effects of diversification and the Keynesian effect of familiarity, we start by considering in Section 3.1 the simple case where the investor may invest in only two risky assets, and the investor is more familiar with the first asset. Then, in Section 3.2 we consider the more general problem where there are $N > 2$ risky assets, and the investor is more familiar with one of these assets. Finally, in Section 3.3 we consider the case of N risky assets, which are divided into M classes, with the investor having a different level of familiarity for each asset class.

3.1 The case with only two risky assets

In the two-asset setting, for the limiting case with no ambiguity about asset returns ($\alpha_n \to 0$),

$$\mu_n = \hat{\mu}_n,$$

"Keynes meets Markowitz" so that the optimal portfolio is given by the familiar Markowitz expression:

$$\pi = \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix} = \frac{1}{\gamma}\Sigma^{-1}\hat{\boldsymbol{\mu}} = \frac{1}{\gamma}\frac{1}{(1-\rho^2)\sigma^2}\begin{bmatrix} \hat{\mu}_1 - \rho\hat{\mu}_2 \\ \hat{\mu}_2 - \rho\hat{\mu}_1 \end{bmatrix}. \tag{11}$$

The portfolio weights above have the well-known properties that investment in the risky assets increases with expected return, μ_S, and decreases with risk aversion, γ, volatility of returns, σ, and the correlation between the returns on the two assets (that is, as ρ goes from -1 to $+1$). In order to focus on the effect of differences in ambiguity across assets, if one sets $\hat{\mu}_1 = \hat{\mu}_2 \equiv \mu$, the above expression simplifies to:[13]

$$\begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix} = \frac{1}{\gamma}\Sigma^{-1}\hat{\boldsymbol{\mu}} = \frac{1}{\gamma}\frac{1}{(1-\rho^2)\sigma^2}\begin{bmatrix} \hat{\mu}(1-\rho) \\ \hat{\mu}(1-\rho) \end{bmatrix}. \tag{12}$$

And for the special case where the two assets are uncorrelated, $\rho = 0$, this reduces to:

$$\pi|_{\rho=0} = \frac{1}{\gamma}\frac{1}{\sigma^2}\begin{bmatrix} \hat{\mu}_1 \\ \hat{\mu}_2 \end{bmatrix} = \frac{1}{\gamma}\frac{1}{\sigma^2}\begin{bmatrix} \hat{\mu} \\ \hat{\mu} \end{bmatrix}. \tag{13}$$

On the other hand, an investor who is averse to ambiguity and has maxmin preferences acts conservatively (or pessimistically) and so in his choice of μ_n, will use its lowest possible estimate within the confidence interval when deciding to go long an asset. Doing the inner minimization for the problem in (9)–(10), we see that the investor will use the following estimates for μ_n:

$$\mu_n = \hat{\mu}_n - \text{sign}(\pi_n)\hat{\sigma}_n\alpha_n, \quad n = \{1, 2\}. \tag{14}$$

This equation implies that if π_n, the weight in Asset n, were positive, then the estimated mean return, $\hat{\mu}_n$, should be reduced downward by the product of α_n, the ambiguity about Asset n, and $\hat{\sigma}_n$, the standard deviation of the estimate $\hat{\mu}_n$. This, of course, would reduce the magnitude of the position in Asset n. In our simplified setting, because the excess return on all the risky assets is positive, the investor will never choose to go short any of the risky assets. In a more general setting, where all assets did not have the same return distribution, it is possible that the investor would like to be long some assets and short other assets. In that case, for assets that the investor would have shorted, the investor will use the *highest* possible estimate of μ_n within the confidence interval when deciding to go short that asset, thereby again *reducing* the magnitude of the position that is taken. Of course, there is also the possibility that if $\hat{\mu_n} > 0$, then after subtracting $\hat{\sigma}_n\alpha_n$ the resulting portfolio weight becomes negative, or that if $\hat{\mu_n} < 0$, then after adding $\hat{\sigma}_n\alpha_n$ the resulting weight is positive. In this case, the optimal portfolio is $\pi_n = 0$.

Substituting (14) back into the problem in (9) gives

$$\max_{\pi} \left\{ \pi^\top\hat{\boldsymbol{\mu}} - \sum_{n}^{N}\text{sign}(\pi_n)\pi_n\hat{\sigma}_n\alpha_n - \frac{\gamma}{2}\pi^\top\Sigma\pi \right\}. \tag{15}$$

[13] We have not canceled out the $(1-\rho)$ in the numerator with the $(1-\rho^2)$ in the denominator in order to allow the reader to see the correspondence with the expressions that appear below.

Then, because

$$\frac{1}{\gamma}\Sigma^{-1}\begin{bmatrix}\hat{\mu}_1-\hat{\sigma}\alpha_1\\ \hat{\mu}_2-\hat{\sigma}\alpha_2\end{bmatrix}=\frac{1}{\gamma}\frac{1}{(1-\rho^2)\sigma^2}\begin{bmatrix}(\hat{\mu}_1-\hat{\sigma}_1\alpha_1)-\rho(\hat{\mu}_2-\hat{\sigma}_2\alpha_2)\\ (\hat{\mu}_2-\hat{\sigma}_2\alpha_2)-\rho(\hat{\mu}_1-\hat{\sigma}_1\alpha_1)\end{bmatrix}, \tag{16}$$

doing the outer maximization over portfolio weights in (15), we see that the optimal portfolio is given by the following proposition,[14] where, in order to focus on the effect of differences in ambiguity across assets, we have assumed that $\hat{\mu}_1=\hat{\mu}_2\equiv\mu$ and where we have assumed, without loss of generality, that $\alpha_1\leq\alpha_2$.

Proposition 1 *In the case where only two risky assets are available, and $\alpha_1\leq\alpha_2$, the optimal portfolio is:*

Case I. If $\frac{\hat{\mu}}{\hat{\sigma}}>\alpha_2+\frac{\rho}{1-\rho}(\alpha_2-\alpha_1)$:

$$\begin{bmatrix}\pi_1\\ \pi_2\end{bmatrix}=\frac{1}{\gamma}\frac{1}{(1-\rho^2)\sigma^2}\begin{bmatrix}\hat{\mu}(1-\rho)-\hat{\sigma}(\alpha_1-\rho\alpha_2)\\ \hat{\mu}(1-\rho)-\hat{\sigma}(\alpha_2-\rho\alpha_1)\end{bmatrix}. \tag{17}$$

Case II. If $\alpha_1<\frac{\hat{\mu}}{\hat{\sigma}}<\alpha_2+\frac{\rho}{1-\rho}(\alpha_2-\alpha_1)$:

$$\begin{bmatrix}\pi_1\\ \pi_2\end{bmatrix}=\begin{bmatrix}\frac{\hat{\mu}-\hat{\sigma}\alpha_1}{\gamma\sigma^2}\\ 0\end{bmatrix}. \tag{18}$$

Case III. If $\alpha_1>\frac{\hat{\mu}}{\hat{\sigma}}$:

$$\begin{bmatrix}\pi_1\\ \pi_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}. \tag{19}$$

We start by studying the simpler situation in which there is zero correlation between the two assets. In Case I above, for the case of zero correlation:

$$\pi|_{\rho=0}=\begin{bmatrix}\pi_1\\ \pi_2\end{bmatrix}=\begin{bmatrix}\frac{\hat{\mu}-\hat{\sigma}\alpha_1}{\gamma\sigma^2}\\ \frac{\hat{\mu}-\hat{\sigma}\alpha_2}{\gamma\sigma^2}\end{bmatrix}. \tag{20}$$

Comparing the expressions in (13) and (20), we see that in the presence of ambiguity about a particular risky asset n, the investment in that risky asset is reduced. Moreover, the investor will hold asset n only if:

$$\hat{\mu}_n>\hat{\sigma}\alpha_n \iff \frac{\hat{\mu}_n}{\hat{\sigma}}>\alpha_n. \tag{21}$$

[14]The proof for this proposition is the same as that for the general case of N risky assets, which is considered below.

Conversely, if the above inequality is not satisfied, then the investor will not hold asset n at all, even though holding this asset would allow the investor to diversify the portfolio. Clearly, as the level of ambiguity α_n increases, the Sharpe ratio that is required so that the investor is willing to hold the asset also increases.

Now studying the case where the two assets are correlated, from the expressions for the optimal portfolio weights in (17) we see that the investor will hold more of Asset 1 than Asset 2 if ambiguity about Asset 1 is less than that for Asset 2: $\alpha_1 < \alpha_2$; that is, the relative investment in these two asset classes depends on the relative ambiguity regarding their return distributions. This is because from Equation (17) we see that if $\alpha_1 < \alpha_2$, it follows that $(\alpha_1 - \rho\alpha_2) < (\alpha_2 - \rho\alpha_1)$, and hence, $\pi_1 > \pi_2 > 0$. Therefore, the portfolio is diversified across the two assets, but has a *tilt* toward the familiar asset. Only when the level of ambiguity in both assets is the same, $\alpha_1 = \alpha_2$, is the proportion of wealth allocated to each asset is the same, $\pi_1 = \pi_2$. Finally, notice that if there is no ambiguity about a particular asset (either $\alpha_1 = 0$ or $\alpha_2 = 0$), the model predicts that it is *always* optimal to take a position in that asset.

Comparing the portfolio weights in the expressions in (12) and (17), we also see that the investor will hold Asset 2 (the unfamiliar asset) only if the ratio $\frac{\hat{\mu}}{\hat{\sigma}}$ for this asset is sufficiently large to offset the effect of its own ambiguity, and the effect that comes from Asset 1's correlation with Asset 2:

$$\frac{\hat{\mu}}{\hat{\sigma}} > \alpha_2 + \frac{\rho}{1-\rho}(\alpha_2 - \alpha_1). \tag{22}$$

In the limit as $\rho \to 1$, the right-hand-side of the above expression goes to infinity, implying that the investor will not hold Asset 2 at all. The reason for this is that Asset 1, the more familiar asset, has a higher ambiguity-adjusted return than the relatively unfamiliar Asset 2 (because $\alpha_1 < \alpha_2$), and as the two assets become increasingly correlated, the diversification benefits from holding Asset 2 diminish. This is a prediction of the model that is consistent with empirical evidence. For instance, during a financial crisis there is an increase in the correlations between returns on different assets, and at such times we often observe investors pulling out of foreign assets into domestic assets because they are more familiar with these assets.

3.2 The case with N risky assets and familiarity about only one asset

We now present the solution for the general case of $N > 2$ assets. Suppose the investor is relatively more familiar with Asset 1 than the other $N-1$ assets. In the setting of our model, this means that the investor has a level of ambiguity α_1 for the mean of the return distribution of Asset 1 and a common level of ambiguity α_{-1} for all the remaining $N-1$ asset in the economy, with $\alpha_1 \leq \alpha_{-1}$.[15] Because the remaining $N-1$ assets are identical in every respect, the optimal portfolio weights in these assets must be identical. We denote by π_1 the portfolio holding in the first asset and by $\pi_{-1} = (\pi_2, \ldots, \pi_N)^\top$ the holdings in the rest of the assets, respectively. The following proposition characterizes the optimal portfolio choice of an ambiguity averse investor.

Proposition 2 *Let α_1 be the level of ambiguity for Asset 1 and α_{-1} the level of ambiguity common across the remaining $N-1$ asset, with $\alpha_1 \leq \alpha_{-1}$. Then the optimal portfolio weights, (π_1, π_{-1}), of an investor who is averse to ambiguity are given by:*

Case I. If $\frac{\hat{\mu}}{\hat{\sigma}} > \alpha_{-1} + \frac{\rho}{1-\rho}(\alpha_{-1} - \alpha_1)$,

$$\pi_1 = \frac{1}{\gamma\sigma^2(1+\rho(N-1))}\left(\hat{\mu} - \hat{\sigma}\frac{(1+\rho(N-2))\alpha_1 - \rho(N-1)\alpha_{-1}}{1-\rho}\right) > 0, \tag{23}$$

$$\pi_{-1} = \frac{1}{\gamma\sigma^2(1+\rho(N-1))}\left(\hat{\mu} - \hat{\sigma}\frac{\alpha_{-1} - \rho\alpha_1}{1-\rho}\right) \cdot \mathbf{1}_{N-1} > 0. \tag{24}$$

Case II. If $\alpha_1 < \frac{\hat{\mu}}{\hat{\sigma}} \leq \alpha_{-1} + \frac{\rho}{1-\rho}(\alpha_{-1} - \alpha_1)$:

$$\pi_1 = \frac{\hat{\mu} - \hat{\sigma}\alpha_1}{\gamma\sigma^2} > 0, \tag{25}$$

$$\pi_{-1} = \mathbf{0}_{N-1}. \tag{26}$$

Case III. If $0 < \frac{\hat{\mu}}{\hat{\sigma}} \leq \alpha_1$:

$$\pi_1 = 0, \tag{27}$$

$$\pi_{-1} = \mathbf{0}_{N-1}. \tag{28}$$

This proposition will allow us to shed some light on the very different views on portfolio choice of Keynes and Markowitz. In Case I, the investor holds both the familiar asset (Asset 1) and the unfamiliar assets (all assets other than Asset 1). But, because $\alpha_1 < \alpha_{-1}$, the weight of Asset 1 in the portfolio exceeds that of the other assets. So, the investor holds familiar assets, as advocated by Keynes, but balances this investment by holding also a portfolio of all the other assets, as advocated by Markowitz.[16] This portfolio is tilted toward familiar assets, and like in the two-asset case, the

[15] As we show in Proposition 6 below, it is possible to generalize the model to account for different degrees of ambiguity on more than two subsets of assets.

[16] Jeremy Siegel, in an article titled "Ben Bernanke's Favorite Stock" that is published on the web site Yahoo!Finance reported that Bernanke held only a single stock, Altria Group (formerly Phillip Morris). Chris Isidore in the article "Bernanke's Bucks" published on CNNMoney.com reports that an overwhelming majority of Bernanke's holdings are in his TIAA-CREF account. Greenspan, on the other hand, has most of his wealth in Treasury bonds and bond funds, with only a small amount invested in individual stocks: besides the investment in General Electric stock held in his wife's 401(k) account, Greenspan's largest stock holdings are in Abbott Laboratories, Kimberly Clark, Anheuser Busch, and H.J. Heinz.

relative investment between the familiar asset and the unfamiliar assets depends on the relative ambiguity regarding the return distributions of these two asset classes, that is, α_1 and α_{-1}.

Case II of Proposition 2 corresponds to the setting where the investor is relatively familiar with a particular asset (that is, $\hat{\mu}/\hat{\sigma} > \alpha_1$) *and* sufficiently unfamiliar with all the other assets (that is, $\frac{\hat{\mu}}{\hat{\sigma}} \leq \alpha_{-1} + \frac{\rho}{1-\rho}(\alpha_{-1} - \alpha_1)$). In this case, the investor follows the advice of Keynes: "the right method in investment is to put fairly large sums into enterprises which one thinks one knows something about .... It is a mistake to think that one limits one's risk by spreading too much between enterprises about which one knows little and has no reason for special confidence." Thus, the household invests only in the familiar risky asset, $\pi_1 > 0$, and not at all in the unfamiliar risky assets, $\pi_{-1} = \mathbf{0}_{N-1}$.

In Case III of Proposition 2, the ambiguity about the expected return of the familiar is large (that is, $\hat{\mu}/\hat{\sigma} \leq \alpha_1$) and the ambiguity about the other assets is even larger, and so $\pi_1 = 0$ and also $\pi_{-1} = \mathbf{0}_{N-1}$, which corresponds to the case of complete non-participation in the stock markets that has been documented empirically.[17] That is, if ambiguity is sufficiently high for all risky assets, the investor will choose not to invest in any of the risky assets, and instead will put all the wealth in the riskfree asset.

To gain further insight on the dichotomy between the views of Keynes and Markowitz on portfolio allocation, we consider in the next proposition the limiting case of an economy with an infinite number of identical securities.

Proposition 3 *Assume investors are less ambiguous about Asset 1 than about the rest of the assets, that is, $\alpha_1 < \alpha_{-1}$. Then, as N, the number of risky assets in the economy, tends to infinity:*

For Case I, in which $\frac{\hat{\mu}}{\hat{\sigma}} > \alpha_{-1} + \frac{\rho}{1-\rho}(\alpha_{-1} - \alpha_1)$, the optimal investment of an ambiguity averse investor in the familiar asset converges to:

$$\lim_{N\to\infty} \pi_1 = \frac{1}{\gamma\sigma_U^2}\hat{\sigma}(\alpha_{-1} - \alpha_1) > 0; \tag{29}$$

the investment in each of the assets with which the investor is not familiar goes to zero:

$$\lim_{N\to\infty} \pi_{-1} = \mathbf{0}_{N-1}; \tag{30}$$

with the total *investment in the $N-1$ assets with which the investor is not familiar being:*

$$\lim_{N\to\infty} (N-1)\pi_{-1} = \frac{1}{\gamma\sigma_S^2(1-\rho)}\left(\hat{\mu}(1-\rho) - \hat{\sigma}(\alpha_{-1} - \rho\alpha_1)\right) \cdot \mathbf{1}_{N-1}. \tag{31}$$

[17] See, for instance, Vissing-Jorgensen (2003) and Campbell (2006).

For Cases II and III of Proposition 2, the portfolio weights do not depend on N, and so even when $N \to \infty$, the expressions for the portfolio weights are the same as those in (25) and (27), respectively.

Propositions 2 and 3 together state that in the presence of ambiguity there is both a concentration in the more familiar assets *and* diversification in other assets. As N approaches infinity, Equation (29) shows that the weight on the more familiar asset approaches a positive constant that depends on the difference in ambiguity about this asset and all the other assets. Equation (30) shows that the *individual* weights on each of the other less-familiar assets approaches zero, with the *total* weight in these $(N-1)$ unfamiliar assets approaching the quantity given in Equation (31). That is, familiarity about Asset 1 implies that the holding in this asset does *not* decrease to zero even as N tends to infinity, while the gain from diversification implies that an investor should hold an infinitesimal amount in each of the other assets.

The concentration in the more familiar asset despite the diversification in the other unfamiliar assets affects also the risk of the portfolio.

Proposition 4 *Assume investors are less ambiguous about Asset 1 than about the rest of the assets, that is, $\alpha_1 < \alpha_{-1}$. For Case I, in which $\frac{\hat{\mu}}{\hat{\sigma}} > \alpha_{-1} + \frac{\rho}{1-\rho}(\alpha_{-1} - \alpha_1)$, as the number of risky assets in the economy grows to infinity, the total risk (variance) ${\pi^*}^\top \Sigma \pi^*$ of the optimal portfolio $\pi^* = (\pi_1^*, \pi_{-1}^*)$ approaches the following quantity:*

$$\lim_{N\to\infty} {\pi^*}^\top \Sigma \pi^* = \frac{\hat{\sigma}^2(\alpha_1 - \alpha_{-1})^2 + 2\hat{\sigma}(\alpha_{-1} - \alpha_1)(\hat{\mu}(1-\rho) - \hat{\sigma}(\alpha_{-1} - \alpha_1)) + \frac{(\hat{\mu}(1-\rho) - \hat{\sigma}(\alpha_{-1}-\alpha_1))^2}{\rho}}{\gamma^2(1-\rho)^2\sigma^2}, \tag{32}$$

where $\hat{\sigma} = \sigma/\sqrt{T}$, $\sigma = \sqrt{\sigma_S^2 + \sigma_U^2}$, and $\rho = \sigma_S^2/\sigma^2$.

As stated in the above proposition, the idiosyncratic risk σ_U^2 remains present in the risk of the portfolio because of concentration. If there was only risk but no ambiguity, then there would not be concentration of investment in Asset 1, and the weight of all assets would approach zero as N tends to infinity, exactly what Markowitz would advocate, as diversification would be the only driving force behind the portfolio choice decision; and in this situation, the risk of the portfolio would be $\hat{\mu}^2/(\gamma^2\sigma_S^2)$, which is driven only by systematic risk.

In the next proposition, we analyze how the optimal portfolio of an ambiguity-averse investor changes with the volatility of the risky assets. We are interested in the *relative* portfolio weights $\omega = \frac{\pi}{\pi^\top \mathbf{1}_N}$. The proposition formalizes the dependence of the relative portfolio weights on the volatility of the assets. As before, we will focus on Case I, in which the optimal portfolio is given by (23)-(24).

Proposition 5 *Assume investors are less ambiguous about Asset 1 than about the rest of the assets, that is, $\alpha_1 < \alpha_{-1}$, and that $\frac{\hat{\mu}}{\hat{\sigma}} > \alpha_{-1} + \frac{\rho}{1-\rho}(\alpha_{-1} - \alpha_1)$. Then,*

1. *If $\frac{1}{N}\alpha_1 + \frac{N-1}{N}\alpha_{-1} < \frac{\mu}{\hat{\sigma}}\frac{\rho N(1+\rho)-(1-\rho^2)}{2\rho N}$, the relative weight ω_1 in the familiar asset decreases with volatility;*
2. *If $\frac{1}{N}\alpha_1 + \frac{N-1}{N}\alpha_{-1} > \frac{\mu}{\hat{\sigma}}\frac{\rho N(1+\rho)-(1-\rho^2)}{2\rho N}$, the relative weight ω_1 in the familiar asset increases with volatility.*

The opposite is true for the relative weight ω_{-1} in the remaining assets.

Proposition 5 provides another characterization of the trade-off between risk and ambiguity. The proposition suggests that the portfolio holding depends in an interesting way on the interaction between volatility and ambiguity. If we interpret the quantity $\frac{1}{N}\alpha_1 + \frac{N-1}{N}\alpha_{-1}$ as a measure of the "average" level of ambiguity in the economy, the proposition shows that when this average measure is low (Item 1), then the portfolio weight in the familiar asset decreases with volatility. On the other hand, if the average ambiguity is high (Item 2), then portfolio holding in the familiar asset *increases* with volatility. Intuitively, this means that, being more familiar about the return on an asset in a highly uncertain environment leads the investor to forego additional diversification and instead concentrate more on the stock about which she knows most. Note, finally that the critical value of the average ambiguity (that is, the expression on the right-hand side of the inequality in Items 1 and 2 is decreasing in σ_U, meaning that, all else being equal, as idiosyncratic volatility increases, Case 2 becomes more likely. Given the recent empirical evidence of an increase of idiosyncratic volatility (see, for example Campbell, Lettau, Malkiel, and Xu (2001)), the results in Proposition 5 suggest potential testable implication of the portfolio choice model in the presence of ambiguity.

3.3 The case with N risky assets and varying familiarity about M asset classes

In the above analysis, we considered the case where the N assets can be divided into two classes with Asset 1 in the first class and the remaining $N-1$ assets in the second class. The characterization of the optimal portfolio weights provided in Proposition 2 takes advantage of this particular feature of the assets. However, investors might not always group assets into just two classes. A case of particular interest for its empirical relevance (Huberman and Jiang (2006)) is the one where investors group the N assets into *three* classes, representing, roughly the case of an investor who places different degree of confidence on the returns estimate of (i) his own-company stock; (ii) a limited set of broadly diversified investment vehicle; and (iii) the remaining set of available assets.

Intuitively, the portfolio problem in which an investors categorizes assets into three separate classes with different degree of ambiguity is not different from the case of two separate asset classes analyzed above. As in the case of two classes of assets, we can fully characterize the optimal portfolio weights for the case of three asset classes. In the next proposition, we characterize the optimal portfolio for an arbitrary number of asset classes about which the investor has varying degrees of familiarity.

Proposition 6 *Let us assume that the set of available N assets can be categorized into $M \leq N$ mutually exclusive asset classes, each containing N_m, $m = 1, \ldots, M$ assets ($N = N_1 + \ldots + N_M$) and each characterized, without loss of generality, by a degree of ambiguity $\alpha_1 < \alpha_2 \leq \ldots \leq \alpha_M$. Denote by $\pi_{\{1,2,\ldots k\}}$, the portfolio weights in each of the $N_1, N_2 \ldots N_k$ assets belonging to classes $1, 2, \ldots, k \leq M$. Define the quantity*

$$\mu^*(m) = \frac{\hat{\sigma}\left((1-\rho)\alpha_m + \rho \sum_{j=1}^{m-1} N_j(\alpha_m - \alpha_j)\right)}{1-\rho} > 0, \quad m = 1, \ldots, M, \tag{33}$$

where $\hat{\sigma} = \sigma/\sqrt{T}$, and the matrix Σ_m, $m = 1, \ldots M$ is defined as

$$\Sigma_m = \sigma^2 \begin{bmatrix} N_1(1+(N_1-1)\rho) & N_1 N_2 \rho & \cdots & N_1 N_m \rho \\ N_2 N_1 \rho & N_2(1+(N_2-1)\rho) & \cdots & N_2 N_m \rho \\ \vdots & \vdots & \ddots & \vdots \\ N_m N_1 \rho & N_k N_2 \rho & \cdots & N_m(1+(N_m-1)\rho) \end{bmatrix}. \tag{34}$$

Then, the optimal portfolio is characterized as follows.

Case I: If $\hat{\mu} > \mu^*(M)$, then:

$$\pi_{\{1,\ldots,M\}} = \frac{1}{\gamma}\Sigma_M^{-1} \begin{bmatrix} N_1(\hat{\mu} - \hat{\sigma}\alpha_1) \\ N_2(\hat{\mu} - \hat{\sigma}\alpha_2) \\ \vdots \\ N_M(\hat{\mu} - \hat{\sigma}\alpha_M) \end{bmatrix}, \tag{35}$$

where

$$\pi_m = \frac{\hat{\mu}(1-\rho) - \hat{\sigma}\left((1-\rho)\alpha_m + \rho\sum_{j\neq m}^{M} N_j(\alpha_m - \alpha_j)\right)}{\gamma\sigma^2(1-\rho)\left(1+\rho(N-1)\right)}, \quad m = 1, \ldots, M. \tag{36}$$

Notice that: $\pi_1 \geq \pi_2 \geq \ldots \geq \pi_M$.

Case II: If $\mu^*(m) < \hat{\mu} \leq \mu^*(m+1)$, $m = 1, \ldots, M-1$, then:

$$\pi_{\{1,\ldots,m\}} = \frac{1}{\gamma}\Sigma_m^{-1}\begin{bmatrix} N_1(\hat{\mu}-\hat{\sigma}\alpha_1) \\ \vdots \\ N_m(\hat{\mu}-\hat{\sigma}\alpha_m) \end{bmatrix}, \tag{37}$$

$$\pi_{\{m+1,\ldots,M\}} = \mathbf{0}_{M-m}, \tag{38}$$

where

$$\pi_i = \frac{\hat{\mu}(1-\rho) - \hat{\sigma}\left((1-\rho)\alpha_i + \rho\sum_{j\neq i}^{m} N_j(\alpha_i-\alpha_j)\right)}{\gamma\sigma^2(1-\rho)\left(1+\rho\left(\sum_{j=1}^{m} N_j - 1\right)\right)}, \quad i = 1,\ldots,m. \tag{39}$$

Case III: If $\hat{\mu} \leq \mu^*(1)$*, then:*

$$\pi_{\{1,\ldots,M\}} = \mathbf{0}_M. \tag{40}$$

In Proposition 2, we saw that for Case II, in which the investor was familiar toward a single asset and sufficiently ambiguous about all the other assets, the optimal portfolio strategy was to invest only in the familiar asset and ignore diversification considerations all together. This was, of course, an extreme portfolio strategy. In the more general setting considered in Proposition 6, we see from Equation (37) that now even in Case II, the investor will invest in not just a single asset but a small number of assets toward which the investor is familiar. And, within the subset of familiar assets, Equation (39) shows that the investor will diversify risk by taking into account the correlation across the subset of those assets in which there is a positive investment.

3.4 Implications of the optimal weights for β and the risk premium

So far, we have looked at the implications of familiarity toward one asset or a few assets for an individual's portfolio. We now study how familiarity toward an asset has an affect at an aggregate level. In order to do so, we consider an economy where there are N firms. We assume that each firm has the same number of employees who all have the same wealth, and that each employee exhibits familiarity toward the stock of the company where she works and the three cases we consider correspond to the three cases described in Proposition 2.

Proposition 7 *Let* α_1 *be the level of ambiguity for Asset 1 and* α_{-1} *the level of ambiguity common across the remaining* $N-1$ *asset, with* $\alpha_1 \leq \alpha_{-1}$*. Then* $\beta_k \equiv \frac{\mathrm{cov}(r_{\mathrm{mkt}}, r_k)}{\mathrm{var}(r_{\mathrm{mkt}})}$ *is given by:*

Case I. If $\frac{\hat{\mu}}{\hat{\sigma}} > \alpha_{-1} + \frac{\rho}{1-\rho}(\alpha_{-1}-\alpha_1)$*, in which case the investor holds both the familiar asset and the rest of the market, then for the case where* N *is large:*

$$\beta_k \approx \frac{\gamma(1-\rho)\rho\sigma^2}{\hat{\mu}(1-\rho)-\hat{\sigma}(\alpha_{-1}-\rho\alpha_1)} = \frac{\gamma(1-\rho)}{\hat{\mu}(1-\rho)-\hat{\sigma}(\alpha_{-1}-\rho\alpha_1)}\sigma_S^2, \tag{41}$$

and if there is no difference in ambiguity across assets:

$$\beta_k \approx \frac{\gamma}{\hat{\mu} - \hat{\sigma}\alpha_1}\sigma_S^2, \tag{42}$$

and if there is no ambiguity at all:

$$\beta_k \approx \frac{\gamma}{\hat{\mu}}\sigma_S^2. \tag{43}$$

Case II. If $\alpha_1 < \frac{\hat{\mu}}{\hat{\sigma}} \leq \alpha_{-1} + \frac{\rho}{1-\rho}(\alpha_{-1} - \alpha_1)$*, in which case the investor holds only the familiar asset and invests zero in all the other assets, then*

$$\beta_k \approx \frac{\gamma(\sigma_S^2 + \sigma_U^2)}{\hat{\mu} - \hat{\sigma}\alpha_1}. \tag{44}$$

Case III. If $0 < \frac{\hat{\mu}}{\hat{\sigma}} \leq \alpha_1$*, in which case no risky assets are held, and so this cannot be a solution in the aggregate.*

Equation (44) shows that for the case where each investor holds only the familiar asset, the β of each asset k with the market depends on both ambiguity and unsystematic volatility. In the case where the investor tilts the portfolio toward the familiar asset but also invests in the rest of the market, Equation (42) shows that while the β depends on ambiguity, it no longer depends on unsystematic volatility. The β for the case where there is no ambiguity is given in Equation (43).

We can also derive the risk-premium for holding risky assets for the three cases described in Proposition 2.

Proposition 8 *Let* α_1 *be the level of ambiguity for Asset 1 and* α_{-1} *the level of ambiguity common across the remaining* $N-1$ *asset, with* $\alpha_1 \leq \alpha_{-1}$*. Then the equilibrium excess return on each risky asset is as follows.*

Case I. If $\frac{\hat{\mu}}{\hat{\sigma}} > \alpha_{-1} + \frac{\rho}{1-\rho}(\alpha_{-1} - \alpha_1)$*, in which case the investor holds both the familiar asset and the rest of the market, then*

$$\hat{\mu} = \gamma\sigma^2\left(1/N + \rho(1 - 1/N)\right) + \hat{\sigma}\alpha_{-1} - \hat{\sigma}(\alpha_{-1} - \alpha_1)/N, \tag{45}$$

and for the case where N *is large:*

$$\hat{\mu} = \gamma\sigma^2\rho + \hat{\sigma}\alpha_{-1}. \tag{46}$$

Case II. If $\alpha_1 < \frac{\hat{\mu}}{\hat{\sigma}} \leq \alpha_{-1} + \frac{\rho}{1-\rho}(\alpha_{-1} - \alpha_1)$*, in which case the investor holds only the familiar asset and invests zero in all the other assets, then the equilibrium return is:*

$$\hat{\mu} = \gamma\sigma^2 + \hat{\sigma}\alpha_1. \tag{47}$$

Case III. If $0 < \frac{\hat{\mu}}{\hat{\sigma}} \leq \alpha_1$*, in which case no risky assets are held, and so this cannot be an equilibrium.*

From Equation (45) we see that, the expected excess return has three components. The first term is the conventional risk premium. The second term is the ambiguity premium due to the ambiguity of the average investor. The third term is the reduction of ambiguity premium due to the familiarity of the investor with a particular asset; this third term is small, however, when N is large.

4 Conclusion

Even though Markowitz's portfolio theory dictates that investors should invest in only a single fund of risky assets, which in equilibrium is the market portfolio, there is substantial evidence that rather than holding just the market portfolio or a well-diversified portfolio, investors hold a substantial amount in just a few assets, often assets with which they are familiar. In this paper we reconcile this apparent contradiction between Markowitz's theory and the empirical evidence by introducing Keynes' view of investment into an otherwise standard Markowitz model.

Our model captures Keynes' view by introducing ambiguity about the true distribution of asset returns and investors' aversion to this ambiguity into the standard Markowitz portfolio-selection setting. The main feature of our model is that it allows investors to distinguish their ambiguity about one asset class relative to others. We show analytically that the model has the following implications, which are consistent with the stylized empirical observations. (i) In the presence of ambiguity about returns on the other assets, the investor holds a disproportionally large amount (relative to Markowitz model) of the familiar asset, but continues to invest in the market portfolio; (ii) The proportion of wealth allocated to the familiar asset increases with an increase in ambiguity about returns of other assets (flight to familiarity); (iii) Investors who are familiar about a particular asset *and* sufficiently ambiguous about all other assets, hold only the familiar asset, as Keynes would have advocated. (iv) Investors who are sufficiently ambiguous about all risky assets, do not participate at all in the equity market. (vi) Finally, the β and the risk premium of assets can depend on both systematic and unsystematic volatility.

A Appendix: Proofs of the Propositions

Proof of Proposition 2

The solution to the inner minimization problem of (9)–(10) is

$$\mu_n = \hat{\mu}_n - \text{sign}(\pi_n)\hat{\sigma}_n\alpha_n, \quad n = 1, \ldots, N, \tag{A1}$$

which, when substituted back into the original problem, gives

$$\max_{\pi} \left\{ \pi^\top \hat{\boldsymbol{\mu}} - \sum_{n}^{N} \text{sign}(\pi_n)\pi_n\hat{\sigma}_n\alpha_n - \frac{\gamma}{2}\pi^\top\Sigma\pi \right\}. \tag{A2}$$

Given our specification of the N stock return process, it must be the case that $\pi_2 = \cdots = \pi_N$. We will denote this common value by π_{-1}. Given that we assume that all risky assets have the same expected return, $\hat{\mu}_1 = \cdots = \hat{\mu}_N = \hat{\mu}$, the original N-dimensional problem simplifies to a two-dimensional problem, with the mean vector $(\hat{\mu}_1, \hat{\mu}_2)$ and variance-covariance matrix

$$\Sigma = \begin{bmatrix} \sigma^2 & (N-1)\rho\sigma^2 \\ (N-1)\rho\sigma^2 & (N-1)[1+(N-2)\rho]\sigma^2 \end{bmatrix} \equiv \begin{bmatrix} \sigma_1^2 & \beta\sigma_1\sigma_2 \\ \beta\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}. \tag{A3}$$

Thus the general problem of N funds reduces to the problem of two stocks with the appropriate mean-return vector and variance-covariance matrix. For the case of two "funds," the expression for the optimal weights is

$$\pi = \frac{1}{\gamma}\Sigma^{-1} \begin{bmatrix} \hat{\mu} - \text{sign}(\pi_1)\hat{\sigma}\alpha_1 \\ (N-1)\left(\hat{\mu} - \text{sign}(\pi_{-1})\hat{\sigma}\alpha_{-1}\right) \end{bmatrix}, \tag{A4}$$

where

$$\Sigma^{-1} = \frac{1}{\sigma_1^2\sigma_2^2(1-\beta^2)} \begin{bmatrix} \sigma_2^2 & -\beta\sigma_1\sigma_2 \\ -\beta\sigma_1\sigma_2 & \sigma_1^2 \end{bmatrix}. \tag{A5}$$

Case I. Suppose that $\hat{\mu}/\hat{\sigma} > (\alpha_{-1} - \rho\alpha_1)/(1-\rho)$. Then,

$$\begin{aligned}
\begin{bmatrix} \pi_1 \\ \pi_{-1} \end{bmatrix} &= \frac{1}{\gamma\sigma_1^2\sigma_2^2(1-\beta^2)} \begin{bmatrix} \sigma_2^2 & -\beta\sigma_1\sigma_2 \\ -\beta\sigma_1\sigma_2 & \sigma_1^2 \end{bmatrix} \begin{bmatrix} \hat{\mu} - \hat{\sigma}\alpha_1 \\ (N-1)(\hat{\mu} - \hat{\sigma}\alpha_{-1}) \end{bmatrix} \\
&= \frac{1}{\gamma(\sigma^2 - \sigma_S^2)(\sigma^2 + (N-1)\sigma_S^2)} \begin{bmatrix} (\hat{\mu} - \hat{\sigma}\alpha_1)(\sigma^2 + (N-2)\sigma_S^2) - (\hat{\mu} - \hat{\sigma}\alpha_{-1})(N-1)\sigma_S^2 \\ (\hat{\mu} - \hat{\sigma}\alpha_{-1})\sigma^2 - (\hat{\mu} - \hat{\sigma}\alpha_1)\sigma_S^2 \end{bmatrix} \\
&= \frac{1}{\gamma\sigma^2(1-\rho)(1+(N-1)\rho)} \begin{bmatrix} \hat{\mu}(1-\rho) - \hat{\sigma}((1+(N-2)\rho)\alpha_1 - (N-1)\rho\alpha_{-1}) \\ \hat{\mu}(1-\rho) - \hat{\sigma}(\alpha_{-1} - \rho\alpha_1) \end{bmatrix}.
\end{aligned} \tag{A6}$$

Because $\alpha_{-1} - \rho\alpha_1 > \alpha_1 - \rho\alpha_{-1} > (1+(N-2)\rho)\alpha_1 - (N-1)\rho\alpha_{-1}$, we have $\pi_1^* > 0$ and $\pi_{-1}^* > 0$. This proves Case I.

Cases II and III. Suppose that $(\rho\alpha_1 - \alpha_{-1})/(1-\rho) \le \hat{\mu}/\hat{\sigma} \le (\alpha_{-1} - \rho\alpha_1)/(1-\rho)$. We start by first proving two intermediate results: (i) it is impossible to have $\pi_{-1}^* < 0$ and $\pi_1^* \ge 0$; and (ii) it is impossible to have $\pi_{-1}^* > 0$ and $\pi_1^* < 0$.

(i) Suppose that $\pi_B = (\pi_1^*, \pi_{-1}^*)$ is the optimal portfolio weight vector with $\pi_{-1}^* < 0$. Then there exists a v such that $\hat{\mu} - \hat{\sigma}v \in [\hat{\mu} - \hat{\sigma}\alpha_1, \hat{\mu} + \hat{\sigma}\alpha_1]$ and

$$\begin{aligned}\pi_B &= \frac{1}{\gamma\sigma_1^2\sigma_2^2(1-\beta^2)}\begin{bmatrix} \sigma_2^2 & -\beta\sigma_1\sigma_2 \\ -\beta\sigma_1\sigma_2 & \sigma_1^2 \end{bmatrix}\begin{bmatrix} \hat{\mu} - \hat{\sigma}v \\ (N-1)(\hat{\mu} + \hat{\sigma}\alpha_{-1}) \end{bmatrix} \\ &= \frac{1}{\gamma\sigma^2(1-\rho)(1+(N-1)\rho)}\begin{bmatrix} \hat{\mu}(1-\rho) - \hat{\sigma}(v(1+(N-2)\rho) + (N-1)\rho\alpha_{-1}) \\ \hat{\mu}(1-\rho) + \hat{\sigma}(\alpha_{-1} + \rho v) \end{bmatrix}.\end{aligned}$$

However,$\hat{\mu}(1-\rho) + \hat{\sigma}(\alpha_{-1} + \rho v) \ge \hat{\mu}(1-\rho) + \hat{\sigma}(\alpha_{-1} - \rho\alpha_1) \ge 0$. This is a contradiction.

(ii) Suppose next that $\pi_1^* < 0$, $\pi_{-1}^* > 0$. Then

$$\begin{aligned}\pi_B &= \frac{1}{\gamma(\sigma^2 - \sigma_S^2)(\sigma^2 + (N-1)\sigma_S^2)}\begin{bmatrix} (\hat{\mu} + \hat{\sigma}\alpha_1)(\sigma^2 + (N-2)\sigma_S^2) - (\hat{\mu} - \hat{\sigma}\alpha_{-1})(N-1)\sigma_S^2 \\ (\hat{\mu} - \hat{\sigma}\alpha_{-1})\sigma^2 - (\hat{\mu} + \hat{\sigma}\alpha_1)\sigma_S^2 \end{bmatrix} \\ &= \frac{1}{\gamma\sigma^2(1-\rho)(1+(N-1)\rho)}\begin{bmatrix} \hat{\mu}(1-\rho) + \hat{\sigma}(\alpha_1(1+(N-2)\rho) + (N-1)\rho\alpha_{-1}) \\ \hat{\mu}(1-\rho) - \hat{\sigma}(\alpha_{-1} + \rho\alpha_1) \end{bmatrix}.\end{aligned}$$

Because $\hat{\mu}(1-\rho) + \hat{\sigma}(\alpha_1(1+(N-2)\rho) + (N-1)\rho\alpha_{-1}) \ge \hat{\mu}(1-\rho) + \hat{\sigma}(\alpha_{-1} - \rho\alpha_1) \ge 0$, we have a contradiction.

We can now prove Cases II and III. From the intermediate results derived in (i) and (ii) above, the only possibilities are either (a) $\pi_{-1}^* = 0$, or (b) $\pi_{-1}^* > 0$ and $\pi_1^* = 0$. Under Possibility (a), the optimal π_1^* is determined by the utility maximization problem

$$\max_{\pi} \min_{\boldsymbol{\mu}} \mu\pi - \frac{\gamma}{2}\pi^2\sigma^2 \quad \text{s.t.} \quad (\mu - \hat{\mu})^2/\hat{\sigma} \le \alpha_1. \tag{A7}$$

And, under Possibility (b), the optimal π_{-1}^* is determined by solving the problem

$$\max_{\pi} \min_{\mu} \mu\pi - \frac{\gamma}{2}\pi^2\sigma^2(1+(N-2)\rho) \quad \text{s.t.} \quad (\mu - \hat{\mu})^2/\hat{\sigma} \le \alpha_{-1}. \tag{A8}$$

Because $\alpha_1 < \alpha_{-1}$,

$$\max_{\pi} \min_{\mu} \mu\pi - \frac{\gamma}{2}\pi^2\sigma^2 > \max_{\pi} \min_{\mu} \mu\pi - \frac{\gamma}{2}\pi^2\sigma^2(1+(N-2)\rho). \tag{A9}$$

Thus, Possibility (b) cannot be optimal. Hence the optimal portfolio weight must correspond to Possibility (a) and the solutions for Cases II and III are:

$$\pi_1^* = \begin{cases} \frac{\hat{\mu} - \hat{\sigma}\alpha_1}{\gamma\sigma^2} & \text{if } \hat{\mu}/\hat{\sigma} > \alpha_1 \\ 0 & \text{if } \hat{\mu}/\hat{\sigma} \le \alpha_1 \end{cases} \tag{A10}$$

$$\pi_{-1}^* = \mathbf{0}_{N-1}. \tag{A11}$$

This completes the proof of the proposition. ∎

Proof of Proposition 3

The limits in (29) and (31) follows immediately from the portfolio weights (23) and (24). ∎

Proof of Proposition 4

The total risk in (32) follows from direct substitution of the optimal portfolio in the expression for the portfolio variance and from taking the limit of the resulting quantity as $N \to \infty$. ∎

Proof of Proposition 5

Using the definitions $\rho = \sigma_S^2/\sigma^2$ and $\hat{\sigma} = \sigma/\sqrt{T}$, we obtain, from Proposition 2,

$$\omega_1 = \frac{\sqrt{T}\mu\left(\sigma^2 - \sigma_S{}^2\right) - \sigma\left((1-N)\alpha_{-1}\sigma S^2 + \alpha_1\left(\sigma^2 + (N-2)\sigma S^2\right)\right)}{\left(N\sqrt{T}\mu - \alpha_1\sigma - N\alpha_{-1}\sigma + \alpha_{-1}\sigma\right)\left(\sigma^2 - \sigma_S^2\right)}. \tag{A12}$$

Differentiating with respect to σ, we obtain

$$\frac{\partial\omega_1}{\partial\sigma} = \frac{(N-1)\left(\alpha_{-1} - \alpha_1\right)A}{\left(N\sqrt{T}\mu - N\sigma\alpha_{-1} + \sigma\alpha_{-1} - \sigma\alpha_1\right)^2\left(\sigma^2 - \sigma_S^2\right)^2}, \tag{A13}$$

where

$$A = 2\left((N-1)\alpha_{-1} + \alpha_1\right)\sigma_S^2\sigma^3 + \sqrt{T}\mu\left(\sigma^4 - (N+2)\sigma_S^2\sigma^2 - (N-1)\sigma_S^4\right). \tag{A14}$$

The derivative (A13) is positive if $A > 0$, that is, if

$$\frac{1}{N}\alpha_1 + \frac{N-1}{N}\alpha_{-1} > \frac{\mu\left(N\rho(\rho+1) - (1-\rho)^2\right)}{2N\rho\hat{\sigma}}. \tag{A15}$$

This concludes the proof of the proposition. ∎

Proof of Proposition 6

The solution to the inner minimization problem of the maxmin problem (9)-(10) is the same as in (A1), which, when substituted back into the original problem, gives the problem in (A2). Note that assets in each subclass will have identical portfolio weight. The case of one asset class is trivial. The optimal portfolio must satisfy

$$\max_{\pi} \pi\left(\hat{\mu} - \text{sign}(\pi)\hat{\sigma}\alpha_1\right) - \frac{\gamma}{2}\pi^2\sigma^2. \tag{A16}$$

The solution π to this problem is

$$\pi = \begin{cases} \frac{1}{\gamma\sigma^2} N_1(\hat{\mu} - \hat{\sigma}\alpha_1) & \text{if } \hat{\mu} > \hat{\sigma}\alpha_1 \\ 0 & \text{if } \hat{\mu} \leq \hat{\sigma}\alpha_1 \end{cases}.$$

We now prove the proposition by induction. That is, assuming that the claim of the proposition is true for the case of N assets with $M-1$ exclusive asset classes with $M \geq 2$, we show that the claim is also true for the case of N assets with M exclusive asset classes.

In general, the maxmin problem has the solution of the form

$$\pi_{\{1,\ldots,M\}} = \frac{1}{\gamma}\Sigma_M^{-1} \begin{bmatrix} N_1(\hat{\mu} - \hat{\sigma}v_1) \\ N_2(\hat{\mu} - \hat{\sigma}v_2) \\ \vdots \\ N_M(\hat{\mu} - \hat{\sigma}v_M) \end{bmatrix}, \tag{A17}$$

with appropriate $v_i \in [-\alpha_i, \alpha_i]$, $i = 1, \ldots, M$, to be determined and its component π_i can be expressed as

$$\pi_i = \frac{\hat{\mu}(1-\rho) - \hat{\sigma}\left((1-\rho)v_i + \rho\sum_{j\neq i}^{M} N_j(v_i - v_j)\right)}{\gamma\sigma^2(1-\rho)\left(1+\rho(N-1)\right)}, \quad i = 1, \ldots, M. \tag{A18}$$

We will make use of this fact in the proof.

First consider the case where $\hat{\mu} > \mu^*(M)$. Then the optimal portfolio weights are:

$$\pi_{\{1,\ldots,M\}} = \frac{1}{\gamma}\Sigma_M^{-1} \begin{bmatrix} N_1(\hat{\mu} - \hat{\sigma}\alpha_1) \\ N_2(\hat{\mu} - \hat{\sigma}\alpha_2) \\ \vdots \\ N_M(\hat{\mu} - \hat{\sigma}\alpha_M) \end{bmatrix}, \tag{A19}$$

provided that $\pi_i > 0$ for all $i = 1, \ldots, M$. We verify that it is indeed the case. By (A18)

$$\pi_i = \frac{\hat{\mu}(1-\rho) - \hat{\sigma}\left((1-\rho)\alpha_i + \rho\sum_{j\neq i}^{M} N_j(\alpha_i - \alpha_j)\right)}{\gamma\sigma^2(1-\rho)\left(1+\rho(N-1)\right)}, \quad i = 1, \ldots, M. \tag{A20}$$

Notice that $\pi_1 \geq \pi_2 \geq \ldots \geq \pi_M$. Since $\hat{\mu} > \mu^*(M)$, $\pi_M > 0$.

Next consider the case where $\mu^*(M-1) < \hat{\mu} \leq \mu^*(M)$. We conjecture that the optimal portfolio weights are given by

$$\pi_{\{1,\ldots,M-1\}} = \frac{1}{\gamma}\Sigma_{M-1}^{-1} \begin{bmatrix} N_1(\hat{\mu} - \hat{\sigma}\alpha_1) \\ \vdots \\ N_{M-1}(\hat{\mu} - \hat{\sigma}\alpha_{M-1}) \end{bmatrix}, \tag{A21}$$

$$\pi_M = 0. \tag{A22}$$

Suppose to the contrary that the optimal $\pi_{\{1,\ldots,M\}}$ in (A17) is such that $\pi_M \neq 0$. There are four subcases to consider. (i) Consider first the subcase where $\pi_M < 0$. Then in (A17), $v_M = -\alpha_M$. Since $\alpha_i \leq \alpha_j$ for $i \leq j$, (A18) implies that $\pi_M > 0$, a contradiction. (ii) Next consider the subcase where $\pi_M > 0$ and there is at least one $j < M$ such that $\pi_j < 0$. Then for the largest j such that $\pi_j < 0$, (A18) again implies that $\pi_j > 0$, a contradiction again. (iii) In the third subcase $\pi_M > 0$ and there is at least one $j < M$ such that $\pi_j = 0$. Fixing π_j at zero, the original maxmin problem reduces to a problem with N asset in $M-1$ classes. By assumption, the claim of the proposition is true for a problem with N asset in $M-1$ classes. But $\pi_M > 0$ is inconsistent with the claim of the proposition for $M-1$ asset classes, a contradiction. (iv) In the fourth subcase, which is the only remaining possibility, all $\pi_i > 0$, $i = 1, \ldots, M$, which implies that in (A17) $v_i = \alpha_i$. But then (A18) and $\hat{\mu} \leq \mu^*(M)$ would imply $\pi_M < 0$, a contradiction. The four subcases together imply that $\pi_M \neq 0$ is impossible. Thus, for the optimal portfolio weight vector, (A22) holds. Fixing π_M at zero, the original maxmin problem reduces to a problem with N assets in $M-1$ classes. Then, by the claim of the proposition, (A21) holds.

Next consider the case where $\mu^*(m) < \hat{\mu} \leq \mu^*(m+1)$, $m = 1, \ldots, M-1$. In this case, $\hat{\mu} \leq \mu^*(M)$ as well. By what is shown above, $\pi_M = 0$ and the problem can therefore be reduced to the case of N assets in $M-1$ class. Thus the claim of the proposition holds. ∎

Proof of Proposition 7

Denote by I_n the number of investors who work for company n, where $n = \{1, \ldots, N\}$. Also, assume that every investor i holds a "symmetric" portfolio of assets n, hence $\pi_{in} = \pi_{kn} = \pi_n$.

Then, the market return is

$$r_{\text{mkt}} = \sum_{n=1}^{N}\sum_{i=1}^{I} \frac{W_i \pi_{in}}{W_{\text{mkt}}} r_n = \sum_{n=1}^{N}\left(\sum_{i=1}^{I_n} \frac{W_i \pi_{in}}{W_{\text{mkt}}} r_n + \sum_{i \notin I_n} \frac{W_i \pi_{in}}{W_{\text{mkt}}} r_n\right), \tag{A23}$$

the covariance of the return on Asset$k$ with the market is

$$\begin{aligned} \text{cov}(r_{\text{mkt}}, r_k) &= \sum_{n=1}^{N}\left(\sum_{i=1}^{I_n} \frac{W_i \pi_{in}}{W_{\text{mkt}}} + \sum_{i \notin I_n} \frac{W_i \pi_{in}}{W_{\text{mkt}}}\right) \text{cov}(r_n, r_k) \\ &= \sum_{n=1}^{N}\left(\frac{\sum_{i=1}^{I_n} W_i}{W_{\text{mkt}}} \pi_n + \frac{\sum_{i \notin I_n} W_i}{W_{\text{mkt}}} \pi_{-n}\right) \text{cov}(r_n, r_k) \end{aligned} \tag{A24}$$

and the variance of the market is

$$\operatorname{cov}(r_{\text{mkt}}, r_{\text{mkt}}) = \sum_{k=1}^{N}\sum_{n=1}^{N}\left(\frac{\sum_{i=1}^{I_n} W_i}{W_{\text{mkt}}}\pi_n + \frac{\sum_{i\notin I_n} W_i}{W_{\text{mkt}}}\pi_{-n}\right)\left(\frac{\sum_{i=1}^{I_k} W_i}{W_{\text{mkt}}}\pi_k + \frac{\sum_{i\notin I_k} W_i}{W_{\text{mkt}}}\pi_{-k}\right)\operatorname{cov}(r_n, r_k). \tag{A25}$$

When all investor have the same wealth,

$$\begin{aligned}
\operatorname{cov}(r_{\text{mkt}}, r_k) &= \sum_{n=1}^{N}\left(\frac{I_n}{I}\pi_n + \frac{I - I_n}{I}\pi_{-n}\right)\operatorname{cov}(r_n, r_k) = \left(\frac{I_1}{I}\pi_1 + \frac{I - I_1}{I}\pi_{-1}\right)[(N-1)\sigma_S^2 + \sigma_S^2 + \sigma_U^2], \\
\operatorname{cov}(r_{\text{mkt}}, r_{\text{mkt}}) &= \sum_{k=1}^{N}\sum_{n=1}^{N}\left(\frac{I_n}{I}\pi_n + \frac{I - I_n}{I}\pi_{-n}\right)\left(\frac{I_k}{I}\pi_k + \frac{I - I_k}{I}\pi_{-k}\right)\operatorname{cov}(r_n, r_k) \\
&= N\left(\frac{I_1}{I}\pi_1 + \frac{I - I_1}{I}\pi_{-1}\right)^2(\sigma_S^2 + \sigma_U^2) + (N-1)N\left(\frac{I_1}{I}\pi_1 + \frac{I - I_1}{I}\pi_{-1}\right)^2\sigma_S^2,
\end{aligned}$$

where we have assumed that $I_i = I_n$. Thus

$$\beta_k = \frac{[N\sigma_S^2 + \sigma_U^2]}{\left(\frac{I_1}{I}\pi_1 + \frac{I-I_1}{I}\pi_{-1}\right)[N(\sigma_S^2 + \sigma_U^2) + (N-1)N\sigma_S^2]}.$$

For I_1/I that is small, which is equivalent to assuming that N is large, $I_1/I = 1/N$:

$$\beta_k = \frac{[N\sigma_S^2 + \sigma_U^2]}{\left(\frac{1}{N}\pi_1 + \frac{N-1}{N}\pi_{-1}\right)[N(\sigma_S^2 + \sigma_U^2) + (N-1)N\sigma_S^2]}.$$

For Case I, we then have that:

$$\pi_{-1} = \frac{\hat{\mu}(1-\rho) - \hat{\sigma}(\alpha_{-1} - \rho\alpha_1)}{\gamma(1-\rho)(1+\rho(N-1))\sigma^2}, \tag{A26}$$

which for large N gives the expression in the proposition:

$$\beta_k \approx \frac{\gamma(1-\rho)\rho\sigma^2}{\hat{\mu}(1-\rho) - \hat{\sigma}(\alpha_{-1} - \rho\alpha_1)} = \frac{\gamma(1-\rho)}{\hat{\mu}(1-\rho) - \hat{\sigma}(\alpha_{-1} - \rho\alpha_1)}\sigma_S^2. \tag{A27}$$

For Case II, the investor holds only the familiar asset. Thus, in aggregate:

$$r_{\text{mkt}} = \sum_{n=1}^{N}\sum_{i=1}^{I}\frac{W_i\pi_{in}}{W_{\text{mkt}}}r_n = \sum_{n=1}^{N}\left(\sum_{i=1}^{I_n}\frac{W_i\pi_{in}}{W_{\text{mkt}}}r_n + \sum_{i\notin I_n}\frac{W_i\pi_{in}}{W_{\text{mkt}}}r_n\right) = \sum_{n=1}^{N}\frac{\left[\sum_{i=1}^{I_n} W_i\right]}{W_{\text{mkt}}}\frac{\hat{\mu} - \hat{\sigma}\alpha_1}{\gamma\sigma^2}r_n. \tag{A28}$$

Then, using the assumption of equal wealth among investors,

$$\operatorname{cov}(r_{\mathrm{mkt}}, r_{\mathrm{mkt}}) = \sum_{k=1}^{N}\sum_{n=1}^{N} \frac{I_n}{I}\frac{I_k}{I}\left(\frac{\hat{\mu}-\hat{\sigma}\alpha_1}{\gamma\sigma^2}\right)^2 \operatorname{cov}(r_n, r_k), \tag{A29}$$

and

$$\operatorname{cov}(r_{\mathrm{mkt}}, r_k) = \sum_{n=1}^{N} \frac{I_n}{I}\left(\frac{\hat{\mu}-\hat{\sigma}\alpha_1}{\gamma\sigma^2}\right) \operatorname{cov}(r_n, r_k), \tag{A30}$$

leading to:

$$\beta_k = \frac{\sum_{n=1}^{N} \frac{I_n}{I}\operatorname{cov}(r_n, r_k)}{\left(\frac{\hat{\mu}-\hat{\sigma}\alpha_1}{\gamma\sigma^2}\right)\sum_{k=1}^{N}\sum_{n=1}^{N} \frac{I_n}{I}\frac{I_k}{I}\operatorname{cov}(r_n, r_k)} = \frac{\sum_{n\neq k}^{N} \frac{I_n}{I}\sigma_S^2 + \frac{I_k I}{(}\sigma_S^2 + \sigma_U^2)}{\left(\frac{\hat{\mu}-\hat{\sigma}\alpha_1}{\gamma\sigma^2}\right)\sum_{k=1}^{N}\sum_{n=1}^{N} \frac{I_n}{I}\frac{I_k}{I}\operatorname{cov}(r_n, r_k)}, \tag{A31}$$

which for large I and small I_k/I gives:

$$\beta_k \approx \frac{\gamma(\sigma_S^2 + \sigma_U^2)}{\hat{\mu}-\hat{\sigma}\alpha_1}, \tag{A32}$$

which is the expression in the proposition. ∎

Proof for Proposition 8

We need to derive the equilibrium of the economy. We study the case where all agents are symmetric. Let π_{i0} denotes agent i's portfolio weight on the riskless asset. Since agents in the economy are symmetric, it must be the case that $\pi_{i0} = \pi_0$ for all i. In equilibrium, since the riskless asset is in zero net supply, π_0 must be equal to zero. Therefore, in equilibrium we must have that $1 = \pi_1 + (N-1)\pi_{-1}$.

We assume that portfolio holding of investors is private information. Now we consider the three cases in Proposition 2.

Case I. The equilibrium condition implies that

$$1 = \frac{N\hat{\mu} - \hat{\sigma}[(N-1)\alpha_{-1} + \alpha_1]}{\gamma(1+\rho(N-1))\sigma^2} = \frac{\hat{\mu}-\hat{\sigma}\alpha_{-1}}{\gamma(1/N + \rho(1-1/N))\sigma^2} + \frac{\hat{\sigma}(\alpha_{-1}-\alpha_1)/N}{\gamma(1/N + \rho(1-1/N))\sigma^2}. \tag{A33}$$

Then, the equilibrium expected return is given by the expression in the proposition.

Case II. The equilibrium condition in this case implies that if $\hat{\mu} > \hat{\sigma}\alpha_1$, then

$$1 = \frac{\hat{\mu}-\hat{\sigma}\alpha_1}{\gamma\sigma^2}. \tag{A34}$$

The equilibrium return is then the expression given in the proposition.

Case III. In this case, all the portfolio weights are strictly negative. This is in contradiction to the condition for equilibrium: $1 = \pi_1 + (N-1)\pi_{-1}$. Thus, this case cannot arise in equilibrium. ∎

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